Higher Mathematics

Course Notes & Solutions

These notes are designed as a revision aid to the Higher Mathematics course, they divide the course into 90 individual lessons. This document should link directly to the PowerPoint 'Higher Course' as well as being backed up by the PowerPoints 'Higher Exam Questions – by Topic', 'Higher Exam Questions – by Paper' and 'Higher Starter Questions'.

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

or $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$

Table of standard integrals:

$f(x)$	$\int f(x)\, dx$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$

Higher Learning Intentions

Unit One

1. The Straight Line (9 Lessons)			
1.1 Straight Lines	(a) I can find and rearrange the equation of a straight line.	5	1 2 3 4 5
	(b) I can calculate the distance between two points.	5	1 2 3 4 5
	(c) I can link the gradient of a straight line and the tangent of an angle.	6	1 2 3 4 5
	(d) I can identify and find perpendicular lines.	6	1 2 3 4 5
	(e) I can identify whether a set of points are collinear.	7	1 2 3 4 5
1.2 Lines in a Triangle	(a) I can find the equation of a median of a triangle.	7	1 2 3 4 5
	(b) I can find the equation of an altitude of a triangle.	8	1 2 3 4 5
	(c) I can find the equation of a perpendicular bisector of a line.	8	1 2 3 4 5
	(d) I can find the point of intersection of two straight lines.	9	1 2 3 4 5

2. Quadratics & Polynomials (8 Lessons)			
2.1 Quadratic Inequalities	(a) I can solve quadratic inequalities.	10	1 2 3 4 5
2.2 Further Quadratics	(a) I can calculate and use the discriminant.	10	1 2 3 4 5
	(b) I can use the discriminant to identify missing coefficients.	11	1 2 3 4 5
	(c) I can express quadratic expressions in the completed square form.	12	1 2 3 4 5
2.3 Polynomial Graphs	(a) I can identify the graph of a quadratic function.	12	1 2 3 4 5
	(b) I can identify the graph of a polynomial function.	13	1 2 3 4 5
2.4 Synthetic Division	(a) I can use synthetic division to factorise a polynomial.	13	1 2 3 4 5
	(b) I can use synthetic division to calculate missing coefficients.	14	1 2 3 4 5

3. Functions & Graphs (6 Lessons)			
3.1 Sets & Functions	(a) I can identify the domain of a function.	15	1 2 3 4 5
	(b) I can identify the range of a function.	15	1 2 3 4 5
	(c) I can find the formula for the composition of two functions.	15	1 2 3 4 5
	(d) I can find the formula for the inverse of a given function.	16	1 2 3 4 5
3.2 Transformations	(a) I can identify and sketch the graphs of transformations.	16	1 2 3 4 5
	(b) I can identify and sketch the inverse of a graph.	17	1 2 3 4 5

4. Differentiation (11 Lessons)			
4.1 Basic Differentiation	(a) I can find the derivative of a basic function.	18	1 2 3 4 5
	(b) I can write a function in differentiable form and differentiate it.	18	1 2 3 4 5
	(c) I can find the derivative of both products and quotients.	18	1 2 3 4 5
	(d) I can calculate the rate of change of a function.	19	1 2 3 4 5
4.2 Tangents	(a) I can use differentiation to calculate the gradient of a curve.	19	1 2 3 4 5
	(b) I can use differentiation to find the equation of a tangent.	19	1 2 3 4 5
4.3 Derivatives & Graphs	(a) I can identify when functions increase, decrease or are stationary.	20	1 2 3 4 5
	(b) I can identify stationary points and their nature.	20	1 2 3 4 5
	(c) I can sketch the graph of a derived function.	21	1 2 3 4 5
4.4 Optimisation	(a) I can optimise a function in a closed interval.	22	1 2 3 4 5
	(b) I can use optimisation to solve practical problems.	22	1 2 3 4 5

Unit Two

5. Trigonometric Equations (5 Lessons)			
5.1 Trig Graphs	(a) I can convert between degrees and radians.	24	1 2 3 4 5
	(b) I can identify exact values in both degrees and radians.	24	1 2 3 4 5
	(c) I can identify the equation of a trigonometric graph.	24	1 2 3 4 5
5.2 Trig Equations	(a) I can solve basic trigonometric equations.	25	1 2 3 4 5
	(b) I can solve trigonometric equations involving factorisation.	25	1 2 3 4 5

6. Vectors (11 Lessons)			
6.1 Vector Calculations	(a) I can find the components of a vector joining two points.	26	1 2 3 4 5
	(b) I can calculate distances in 3 dimensions and magnitudes.	26	1 2 3 4 5
	(c) I can identify vector pathways.	26	1 2 3 4 5
	(d) I can identify and find unit vectors.	27	1 2 3 4 5
6.2 Vectors & Ratio	(a) I can identify and work with collinear points.	27	1 2 3 4 5
	(b) I can find the ratio in which a point divides a line.	28	1 2 3 4 5
	(c) I can use a ratio to identify a point on a line.	28	1 2 3 4 5
6.3 Scalar Product	(a) I can calculate the scalar product of two vectors.	29	1 2 3 4 5
	(b) I can calculate an angle using the scalar product.	30	1 2 3 4 5
	(c) I can identify perpendicular vectors using the scalar product.	30	1 2 3 4 5
	(d) I can use the commutative and distributive laws with vectors.	31	1 2 3 4 5

7. Exponentials & Logarithms (7 Lessons)			
7.1 Laws of Logarithms	(a) I can evaluate logarithmic expressions.	32	1 2 3 4 5
	(b) I can use the laws of logarithms to simplify expressions.	32	1 2 3 4 5
7.2 Logarithmic Equations	(a) I can solve equations involving logarithms.	32	1 2 3 4 5
	(b) I can solve equations involving logarithms and factorisation.	33	1 2 3 4 5
7.3 Exponential Growth	(a) I can solve problems on exponential growth and decay.	33	1 2 3 4 5
7.4 Logarithmic Graphs	(a) I can work with exponential and logarithmic graphs.	34	1 2 3 4 5
	(b) I can solve problems on experimental data resulting in graphs.	34	1 2 3 4 5

8. Integration (7 Lessons)			
8.1 Basic Integration	(a) I can use the basic rules of integration.	35	1 2 3 4 5
	(b) I can use the basic rules of integration with products and quotients.	35	1 2 3 4 5
	(c) I can evaluate a definite integral.	35	1 2 3 4 5
8.2 Integration & Area	(a) I can calculate the area of a curve above the x-axis.	36	1 2 3 4 5
	(b) I can calculate the area of a curve both above and below the x-axis.	36	1 2 3 4 5
	(c) I can calculate the area between two curves.	37	1 2 3 4 5
8.3 Differential Equations	(a) I can evaluate the constant of integration in a differential equation.	38	1 2 3 4 5

Unit Three

9. Trigonometric Formulae (9 Lessons)			
9.1 Addition Formulae	(a) I can use addition formulae with exact values.	39	1 2 3 4 5
	(b) I can use addition formulae with triangles.	39	1 2 3 4 5
9.2 Double-Angles	(a) I can use the double-angle formulae with exact values.	39	1 2 3 4 5
	(b) I can use the double-angle formulae with triangles.	40	1 2 3 4 5
9.3 Wave Function	(a) I can write combinations of sine and cosine as single functions.	41	1 2 3 4 5
	(b) I can sketch the graph of a combination of two functions.	41	1 2 3 4 5
9.4 Trig Applications	(a) I can solve equations involving the double-angle formulae.	42	1 2 3 4 5
	(b) I can solve equations involving the wave function.	43	1 2 3 4 5
	(c) I can use trigonometric identities in proofs.	43	1 2 3 4 5

10. Recurrence Relations (5 Lessons)			
10.1 Terms in a Sequence	(a) I can evaluate the next term in a sequence.	44	1 2 3 4 5
	(b) I can evaluate the previous term in a sequence.	44	1 2 3 4 5
	(c) I can find the formula of a recurrence relation from its sequence.	44	1 2 3 4 5
10.2 Limit of a Sequence	(a) I can identify whether a sequence has a limit and calculate this.	45	1 2 3 4 5
	(b) I can set up and solve recurrence relations in context.	45	1 2 3 4 5

11. The Circle (7 Lessons)			
11.1 Centre & Radius	(a) I can find the equation of a circle given its centre and radius.	47	1 2 3 4 5
11.2 Equation of a Circle	(a) I can work with the general equation of a circle.	47	1 2 3 4 5
	(b) I can identify whether an equation represents a circle.	47	1 2 3 4 5
11.3 Lines & Circles	(a) I can find the equation of a tangent to a circle.	48	1 2 3 4 5
	(b) I can find the point(s) of intersection of a straight line and a circle.	48	1 2 3 4 5
	(c) I can work with circles and tangents.	49	1 2 3 4 5
11.4 Intersecting Circles	(a) I can identify the type of intersection between two circles.	49	1 2 3 4 5

12. Further Calculus (5 Lessons)			
12.1 Further Derivatives	(a) I can differentiate functions using the chain rule.	51	1 2 3 4 5
	(b) I can differentiate functions involving sine and cosine.	51	1 2 3 4 5
	(c) I can differentiate functions involving sine, cosine and chain rule.	51	1 2 3 4 5
12.2 Further Integrals	(a) I can integrate composite functions.	52	1 2 3 4 5
	(b) I can integrate functions involving sine and cosine.	52	1 2 3 4 5

1. The Straight Line

1.1 Straight Lines

1.1 (a) I can find and rearrange the equation of a straight line.

From National 5: You should know the following:

The **gradient** of a straight line passing through the points (x_1, y_1) and (x_2, y_2) is found using: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

- A line passing through (a, b) with gradient m has equation: $y - b = m(x - a)$.
- A **horizontal** line $y = a$ has a gradient of zero.
- A **vertical** line $x = b$ has an undefined gradient.
- Two lines are **parallel** if they have the **same** gradient.

If you **rearrange** a straight line equation to the form $y = mx + c$ you can identify its **gradient** and **y-intercept.**

In Higher: The examples of this can be more challenging.

A straight-line AB has equation $4y + 5x = 8$. What is the gradient of any straight line parallel to AB?

$4y = -5x + 8$

$y = -\frac{5}{4}x + 2$

$Therefore\ m = -\frac{5}{4}.$

Find the equation of the line passing through (3,-6) which is parallel to the line with equation $y + 6x - 8 = 0$.

$y = -6x + 8$

$Therefore\ m = -6.$

$y - -6 = -6(x - 3)$

$y + 6 = -6x + 18$

$y = -6x + 12$

Find the equation of the line passing through (-2, 1) which is parallel to the line with equation $4y + 3x = 12$.

$4y = -3x + 12$

$y = -\frac{3}{4}x + 3$

$Therefore\ m = -\frac{3}{4}.$

$y - 1 = -\frac{3}{4}(x - -2)$

$4y - 4 = -3x - 6$

$4y = -3x - 2$

$y = -\frac{3}{4}x - \frac{1}{2}$

C and D are the points (-3, 2) and (6, a). CD is parallel to the line $y - 2x = 10$. Determine the value of a.

$y = 2x + 10$

$Therefore\ m = 2.$

$y - 2 = 2(x - -3)$

$y - 2 = 2x + 6$

$y = 2x + 8$

$At\ (6, a)\ x = 6, y = a.$

$a = 2(6) + 8$

$= 20$

1.1 (b) I can calculate the distance between two points.

The **distance** between two points (x_1, y_1) and (x_2, y_2) can be found using the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Calculate the distance between (1, 6) and (-5, 2).

$d = \sqrt{6^2 + 4^2}$

$= \sqrt{52}$

Calculate the distance between (2, -5) and (7, 7).

$d = \sqrt{5^2 + 12^2}$

$= 13$

Calculate the distance between (2, 1) and (-3, 5).

$d = \sqrt{5^2 + 4^2}$

$= \sqrt{41}$

Calculate the distance between (0, -3) and (-5, 2).

$d = \sqrt{5^2 + 5^2}$

$= \sqrt{50}$

Calculate the distance between (-6, 1) and (-2, -2).

$d = \sqrt{4^2 + 3^2}$

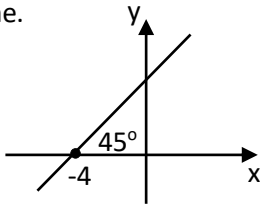
$= 5$

1.1 (c) I can link the gradient of a straight line and the tangent of an angle.

The gradient of a straight line can be calculated as the tangent of the angle (θ) between the straight line and the **positive direction** of the x-axis.

$$m = \tan\theta$$

Find the equation of the line.



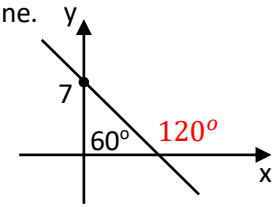
$m = \tan 45^o$

$= 1$

$y - 0 = 1(x - -4)$

$y = x + 4$

Find the equation of the line.



$m = \tan 120^o$

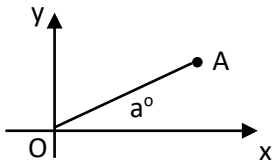
$= -\sqrt{3}$

$y - 7 = -\sqrt{3}(x - 0)$

$y - 7 = -\sqrt{3}x$

$y = -\sqrt{3}x + 7$

The line OA has gradient 0.3. Find the size of a°.



$m = \tan a^o$

$0.3 = \tan a^o$

$\tan^{-1}(0.3) = a^o$

$16.70^o = a^o$

A line has equation $y - 2x = 7$.

Determine the angle this line makes with the positive direction of the x-axis.

$y = 2x + 7$

$Therefore\ m = 2.$

$m = \tan\theta$

$2 = \tan\theta$

$\tan^{-1}(2) = \theta$

$63.43^o = \theta$

1.1 (d) I can identify and find perpendicular lines.

Two lines with gradients m_1 and m_2 are **perpendicular** if they meet at 90°, this is the case if:

$$m_1 \times m_2 = -1$$

If you know the gradient of one line and want to find the gradient of a line **perpendicular** to this:

1. Flip the fraction upside down.
2. Change the sign.

Note: Lines with undefined gradients are perpendicular to those with gradients of 0, so x = 3 and y = 4 are perpendicular.

Are the lines with equations $y - 7 = 2x$ and $6y = 2 - 3x$ perpendicular?

$y = 2x + 7$

$Therefore\ m_1 = 2.$

$y = -\frac{1}{2}x + \frac{1}{3}$

$Therefore\ m_2 = -\frac{1}{2}.$

$m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right)$

$= -1$

$So\ perpendicular.$

The line $2y = 5x + 7$ is perpendicular to the line with equation $y = kx$. What is the value of k?

$y = \frac{5}{2}x + \frac{7}{2}$

$Therefore\ m = \frac{5}{2}.$

$k = -\frac{2}{5}$

Find the equation of the line passing through (2, 4) and perpendicular to the line $5x + y - 2 = 0$.

$y = -5x + 2$

$Therefore\ m = -5.$

$m_\perp = \frac{1}{5}$

$y - 4 = \frac{1}{5}(x - 2)$

$5y - 20 = x - 2$

$5y = x + 18$

$y = \frac{1}{5}x + \frac{18}{5}$

AB is perpendicular to the line $x + y = 8$. A has coordinates (5, -3). Find the equation of AB.

$y = -x + 8$

$Therefore\ m = -1.$

$m_\perp = 1$

$y - -3 = 1(x - 5)$

$y + 3 = x - 5$

$y = x - 8$

1.1 (e) I can identify whether a set of points are collinear.

Points which lie on the **same straight line** are said to be **collinear**.

Two lines are **collinear** if they have the **same gradient** and a **point in common**.

Prove that A(-4, -5), B(0,-3) and C(8, 1) are collinear.

$m_{AB} = \frac{-3 - -5}{0 - -4}$

$= \frac{1}{2}$

$m_{BC} = \frac{1 - -3}{8 - 0}$

$= \frac{1}{2}$

Therefore $m_{AB} = m_{BC}$ and they have point B in common therefore A, B and C are collinear.

Are the points F(-3, 2), G(4,1) and H(7, 8) collinear?

$m_{FG} = \frac{1 - 2}{4 - -3}$

$= -\frac{1}{7}$

$m_{GH} = \frac{8 - 1}{7 - 4}$

$= \frac{7}{3}$

Therefore $m_{FG} \neq m_{GH}$ and F, G and H are not collinear.

L, M and N are points such that LM is parallel to the line with equation $y + 2x = 0$ and MN makes an angle of 135°. with the positive direction of the x-axis.

Are L, M and N collinear?

$y = -2x$

Therefore $m_{LM} = -2$.

$m_{MN} = \tan 135^o$

$= -1$

Therefore $m_{LM} \neq m_{MN}$ and L, M and N are not collinear.

A, B and C are points such that AB is parallel to the line with equation $y - 3x = 4$ and BC makes an angle of 60°. with the positive direction of the x-axis.

Are A, B and C collinear?

$y = 3x + 4$

Therefore $m_{AB} = 3$.

$m_{BC} = \tan 60^o$

$= \sqrt{3}$

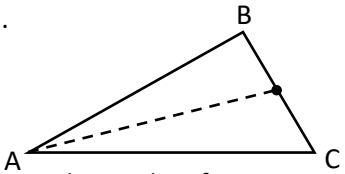
Therefore $m_{AB} \neq m_{BC}$ and A, B and C are not collinear.

1.2 Lines in a Triangle

1.2 (a) I can find the equation of a median of a triangle.

The **median** of a triangle is a line from a **vertex** to the **midpoint** of the opposite side.

1. Find the **midpoint** of the side opposite the given point.
2. Find the **gradient** of the line joining the given point and the midpoint.
3. Substitute into $y - b = m(x - a)$.



The median from A.

To verify if a point lies on a line **substitute** in the x-coordinate and show this gives the y-coordinate (or vice versa).

Find the median from D for a triangle with vertices: D(4, 5), E(8, -2) and F(2, 0).

$M = midpt\ EF = (5, -1)$

$m_{DM} = \frac{-1 - 5}{5 - 4}$

$= -6$

$y - 5 = -6(x - 4)$

$y - 5 = -6x + 24$

$y = -6x + 29$

Find the median from Q for a triangle with vertices: P(2, 8), Q(5, 8) and R(4, -2).

$M = midpt\ PR = (3, 3)$

$m_{QM} = \frac{3 - 8}{3 - 5}$

$= \frac{5}{2}$

$y - 8 = \frac{5}{2}(x - 5)$

$2y - 16 = 5x - 25$

$2y = 5x - 9$

$y = \frac{5}{2}x - \frac{9}{2}$

Find the median from A for a triangle with vertices: A(2, 6), B(7, 4) and C(-3, 0).

$M = midpt\ BC = (2, 2)$

$m_{AM} = \frac{2 - 6}{2 - 2}$

$= -\frac{4}{0}\ undefined$

$x = 2$

Verify that (4, 5) lies on the median $y = 3x - 7$.

$x = 4 \Rightarrow y = 3(4) - 7$

$= 5$

Therefore it lies on the median.

Does (2, 3) lie on the median $y + 4x = 11$?

$x = 2 \Rightarrow y + 4(2) = 11$

$y = 3$

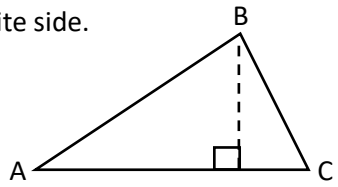
Therefore it lies on the median.

1.2 (b) I can find the equation of an altitude of a triangle.

The **altitude** of a triangle is a line from a **vertex** which is **perpendicular** to the opposite side.

1. Find the **gradient** of the side opposite to the given point.
2. Find the gradient **perpendicular** to this.
3. Substitute into $y - b = m(x - a)$.

The altitude from B.



Find the altitude from B for a triangle with vertices: B(1, 6), C(7, 3) and D(-5, -1).

$$m_{CD} = \frac{-1-3}{-5-7}$$
$$= \frac{1}{3}$$
$$m_{\perp} = -3$$
$$y - 6 = -3(x - 1)$$
$$y - 6 = -3x + 3$$
$$y = -3x + 9$$

Find the altitude from Q for a triangle with vertices: P(4, 1), Q(2, 3) and R(8, -2).

$$m_{PR} = \frac{-2-1}{8-4}$$
$$= -\frac{3}{4}$$
$$m_{\perp} = \frac{4}{3}$$
$$y - 3 = \frac{4}{3}(x - 2)$$
$$3y - 9 = 4x - 8$$
$$3y = 4x + 1$$
$$y = \frac{4}{3}x + \frac{1}{3}$$

Find the altitude from F for a triangle with vertices: E(3, 2), F(7, 3) and G(3, -4).

$$m_{EG} = \frac{-4-2}{3-3}$$
$$= \frac{-6}{0} \; undefined$$
$$m_{\perp} = 0$$
$$y - 3 = 0(x - 7)$$
$$y - 3 = 0$$
$$y = 3$$

Verify that (3, -1) lies on the altitude $y = 2x - 7$.

$$x = 3 \Rightarrow y = 2(3) - 7$$
$$= -1$$

Therefore it lies on the altitude.

Does (6, 4) lie on the altitude $2x - 4y = -8$?

$$x = 6 \Rightarrow 2(6) - 4y = -8$$
$$-4y = -20$$
$$y = 5$$

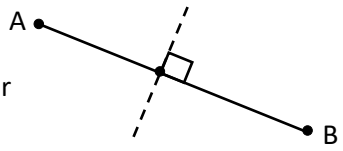
Therefore it lies on the altitude.

1.2 (c) I can find the equation of a perpendicular bisector of a line.

The **perpendicular bisector** of a line is another line passing through its **midpoint** and **perpendicular** to it.

1. Find the **gradient** of the line joining the given points.
2. Find the gradient **perpendicular** to this.
3. Find the coordinates of the **midpoint**.
4. Substitute into $y - b = m(x - a)$.

The perpendicular bisector of AB.



Find the perpendicular bisector of the line joining (6, -4) and (-2, -2).

$$m = \frac{-2--4}{-2-6}$$
$$= -\frac{1}{4}$$
$$m_{\perp} = 4$$
$$M = midpt = (2, -3)$$
$$y - -3 = 4(x - 2)$$
$$y + 3 = 4x - 8$$
$$y = 4x - 11$$

Find the perpendicular bisector of the line joining K(3, -2) and L (5, 10).

$$m_{KL} = \frac{10--2}{5-3}$$
$$= 6$$
$$m_{\perp} = -\frac{1}{6}$$
$$M = midpt\ KL = (4, 4)$$
$$y - 4 = -\frac{1}{6}(x - 4)$$
$$6y - 24 = -x + 4$$
$$6y = -x + 28$$
$$y = -\frac{1}{6}x + \frac{14}{3}$$

Find the perpendicular bisector of the line joining (8, 5) and (-2, 5).

$$m = \frac{5-5}{-2-8}$$
$$= 0$$
$$m_{\perp}\ undefined$$
$$M = midpt = (3,5)$$
$$x = 3$$

Verify that the perpendicular bisector $2x + 5y = -13$ cuts the midpoint of the line AB where A(4, 0) and B(-2, -6).

$$M = midpt = (1, -3)$$
$$x = 1 \Rightarrow$$
$$2(1) + 5y = -13$$
$$5y = -15$$
$$y = -3$$

Therefore it cuts the midpoint of the line.

1.2 (d) I can find the point of intersection of two straight lines.

When you find the equations of two straight lines you can find the coordinates of their **point of intersection** by treating them as a pair of **simultaneous equations**.

Find the point of intersection of the perpendicular bisector $y = 2x - 14$ and the median $y = -3x + 16$.

$2x - 14 = -3x + 16$

$5x = 30$

$x = 6$

$x = 6 \Rightarrow y = 2(6) - 14$

$= -2$

$Coordinates\ (6, -2).$

Find the point of intersection of the perpendicular bisector $y = 4x - 2$ and the median $y = -5x + 25$.

$4x - 2 = -5x + 25$

$9x = 27$

$x = 3$

$x = 3 \Rightarrow y = 4(3) - 2$

$= 10$

$Coordinates\ (3,\ 10).$

Find the point of intersection of the altitude $3x + 2y = 5$ and the perpendicular bisector $y - 4x = -14$.

$y = 4x - 14$

$2y = -3x + 5$

$y = -\frac{3}{2}x + \frac{5}{2}$

$4x - 14 = -\frac{3}{2}x + \frac{5}{2}$

$8x - 28 = -3x + 5$

$11x = 33$

$x = 3$

$x = 3 \Rightarrow y = 4(3) - 14$

$= -2$

$Coordinates\ (3, -2).$

Find the point of intersection of the median $2x + y = 1$ and the altitude $3y - 5x = 25$.

$y = -2x + 1$

$3y = 5x + 25$

$y = \frac{5}{3}x + \frac{25}{3}$

$-2x + 1 = \frac{5}{3}x + \frac{25}{3}$

$-6x + 3 = 5x + 25$

$-22 = 11x$

$-2 = x$

$x = -2 \Rightarrow$

$y = -2(-2) + 1$

$= 5$

$Coordinates\ (-2,\ 5).$

2. Quadratics & Polynomials

2.1 Quadratic Inequalities

2.1 (a) I can solve quadratic inequalities.

A **polynomial** is a mathematical expression of the form: $ax^n + bx^{n-1} + cx^{n-2} + \cdots + dx + e$.

A **quadratic** is a **polynomial** of order 2 i.e. $ax^2 + bx + c$.

From National 5: You should know the following:

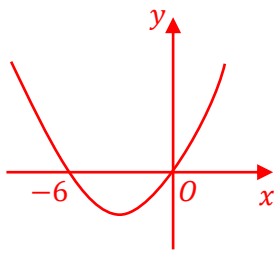
The roots of a **quadratic equation** $ax^2 + bx + c = 0$ can be found using **factorisation**.

In Higher: You may be asked to solve **quadratic inequalities**, to do so:

1. **Rearrange** the inequality to get $ax^2 + bx + c$ on one side and zero on the other.
2. Find the roots of the quadratic inequality using **factorisation**.
3. **Sketch** the graph of this.
4. Determine the **region(s)** either above or below the x-axis.

Solve $x^2 + 6x > 0$.

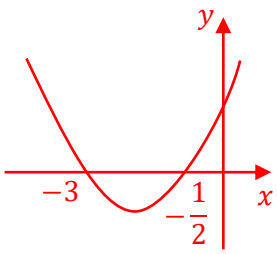
$x(x + 6) > 0$



$So\ x < -6,\ x > 0.$

Solve $2x^2 + 7x + 3 < 0$.

$(2x + 1)(x + 3) < 0$

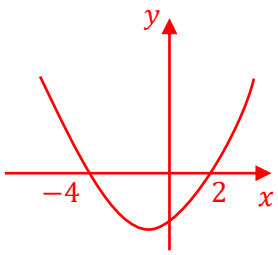


$So\ -3 < x < -\frac{1}{2}.$

Solve $8 - 2x - x^2 > 0$.

$x^2 + 2x - 8 < 0$

$(x + 4)(x - 2) < 0$

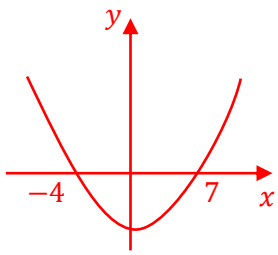


$So\ -4 < x < 2.$

Solve $x^2 - 3x > 28$.

$x^2 - 3x - 28 > 0$

$(x + 4)(x - 7) < 0$



$So\ x < -4, x > 7.$

2.2 Further Quadratics

2.2 (a) I can calculate and use the discriminant.

From National 5: You should know the following:

If a **quadratic equation** $ax^2 + bx + c = 0$ **does not factorise** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can find the **roots**.

The **discriminant**, $b^2 - 4ac$, can be used to determine the **nature of the roots** of a quadratic equation:

- If $b^2 - 4ac > 0$ then the quadratic equation has **two real distinct roots**.
- If $b^2 - 4ac = 0$ then the quadratic equation has **equal real roots**.
- If $b^2 - 4ac < 0$ then the quadratic equation has **no real roots**.

In Higher: This is formalised to the following:

- If $b^2 - 4ac$ is a **square** number, roots are **rational** and $ax^2 + bx + c$ will factorise.
- If $b^2 - 4ac$ is **not a square** number, roots are **irrational** and $ax^2 + bx + c$ will not factorise.
- If $b^2 - 4ac = 0$ this means that the (repeated) root is a **tangent** to the line.

By **equating** them the same process can help you decide **if and how a line and a parabola intersect**.

Determine the nature of the roots of the quadratic equations:

$2x^2 + 7x + 9.$

$b^2 - 4ac = 7^2 - 4(2)(9)$

$= -23$

So no real roots.

$x^2 + 7x = -7.$

$x^2 + 7x + 7 = 0$

$b^2 - 4ac = 7^2 - 4(1)(7)$

$= 21$

So two real distict roots.

Are the roots of the following real and rational?

Discriminant 17.

Two real distinct roots.

Discriminant is not a square number therefore the roots are irrational.

Discriminant 49.

Two real distinct roots.

Discriminant is a square number therefore the roots are rational.

Show the line $y = 2x - 3$ does not intersect the parabola with equation $y = x^2 + 3x + 5.$

$x^2 + 3x + 5 = 2x - 3$

$x^2 + x + 8 = 0$

$b^2 - 4ac = 1^2 - 4(1)(8)$

$= -31$

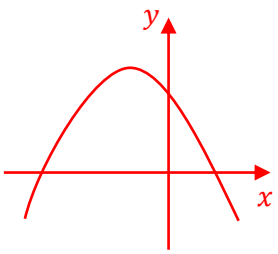
So no real roots.

Therefore they do not intersect.

If $y = ax^2 + bx + c$ and $a < 0$ and $b^2 - 4ac > 0$ sketch a possible graph of the equation.

$a < 0$ so $\cap$ shape.

$b^2 - 4ac > 0$ so two real distinct roots.



2.2 (b) I can use the discriminant to identify missing coefficients.

You may be told the **nature of the roots** in order to use the discriminant to create either an equation or inequation containing **unknown coefficients** which you can solve.

The roots of the following are equal. Determine the values for k and m.

$kx^2 + 4x - 4 = 0.$

Equal real roots when $b^2 - 4ac = 0.$

$4^2 - 4(k)(-4) = 0$

$16 + 16k = 0$

$16k = -16$

$k = -1$

$(m-1)x^2 + 8x + 1 = 0.$

Equal real roots when $b^2 - 4ac = 0.$

$8^2 - 4(m-1)(1) = 0$

$68 - 4m = 0$

$68 = 4m$

$17 = m$

Find the values of p such that the following have no real roots.

$x^2 - 6x + p = 0.$

No real roots when $b^2 - 4ac < 0.$

$(-6)^2 - 4(1)(p) < 0$

$36 - 4p < 0$

$36 < 4p$

$9 < p$

$x^2 + 4x + 2 - p = 0.$

No real roots when $b^2 - 4ac < 0.$

$4^2 - 4(1)(2-p) < 0$

$8 + 4p < 0$

$4p < -8$

$p < -2$

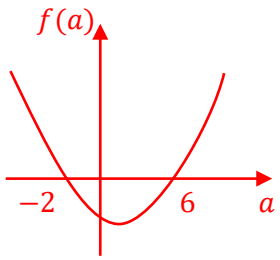
Find the range of values of a such that $x^2 + (a-2)x + 4$ has two real distinct roots.

Two real distinct roots when $b^2 - 4ac > 0.$

$(a-2)^2 - 4(1)(4) > 0$

$a^2 - 4a - 12 > 0$

$(a+2)(a-6) > 0$



So $a < -2, a > 6.$

Find the range of values for k such that $kx^2 + 6x + 4k = 0$ has real roots.

Real roots when $b^2 - 4ac \geq 0.$

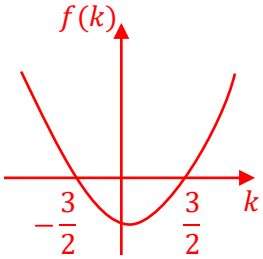
$6^2 - 4(k)(4k) \geq 0$

$36 - 16k^2 \geq 0$

$16k^2 - 36 \leq 0$

$4(4k^2 - 9) \leq 0$

$4(2k+3)(2k-3) \leq 0$



So $-\frac{3}{2} \leq k \leq \frac{3}{2}.$

2.2 (c) I can express quadratic expressions in the completed square form.

From National 5: You should know the following:

You can write $x^2 + bx + c$ in the **completed square form** $(x + p)^2 + q$.

In Higher: You may be asked to write $ax^2 + bx + c$ in the form $a(x + p)^2 + q$.

$x^2 + 8x + 3.$

$= (x + 4)^2 - 16 + 3$

$= (x + 4)^2 - 13$

$x^2 - 18x - 4.$

$= (x - 9)^2 - 81 - 4$

$= (x - 9)^2 - 85$

$x^2 - 6x + 82.$

$= (x - 3)^2 - 9 + 82$

$= (x - 3)^2 + 73$

$2x^2 - 12x - 4.$

$a(x + p)^2 + q$

$= ax^2 + 2apx + ap^2 + q$

$* \ a = 2$

$* \ 2ap = -12$

$4p = -12$

$p = -3$

$* \ ap^2 + q = -4$

$18 + q = -4$

$q = -22$

$= 2(x - 3)^2 - 22$

$4x^2 - 24x - 7.$

$a(x + p)^2 + q$

$= ax^2 + 2apx + ap^2 + q$

$* \ a = 4$

$* \ 2ap = -24$

$8p = -24$

$p = -3$

$* \ ap^2 + q = -7$

$36 + q = -7$

$q = -43$

$= 4(x - 3)^2 - 43$

$-3x^2 - 6x + 37.$

$a(x + p)^2 + q$

$= ax^2 + 2apx + ap^2 + q$

$* \ a = -3$

$* \ 2ap = -6$

$-6p = -6$

$p = 1$

$* \ ap^2 + q = 37$

$-3 + q = 37$

$q = 40$

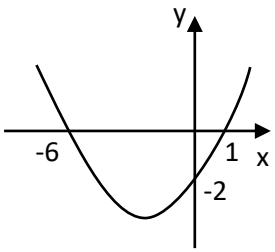
$= -3(x + 1)^2 + 40$

2.3 Polynomial Graphs

2.3 (a) I can identify the graph of a quadratic function.

It is possible to identify the equation of a **quadratic function** from its graph by:

- Using its **roots** to identify **factors**
- Using another point to calculate the value of the **constant**.
- Look out for any repeated (equal) roots!



$y = k(x + 6)(x - 1)$

$When \ x = 0, \ y = -2.$

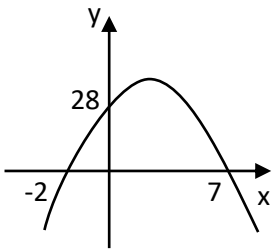
$-2 = k(0 + 6)(0 - 1)$

$-2 = -6k$

$\frac{1}{3} = k$

$Therefore$

$y = 3(x + 3)(x - 1).$



$y = k(x + 2)(x - 7)$

$When \ x = 0, \ y = 28.$

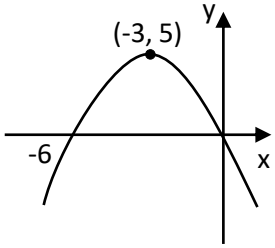
$28 = k(0 + 2)(0 - 7)$

$28 = -14k$

$-2 = k$

$Therefore$

$y = -2(x + 2)(x - 7).$



$y = kx(x + 6)$

$When \ x = -3, \ y = 5.$

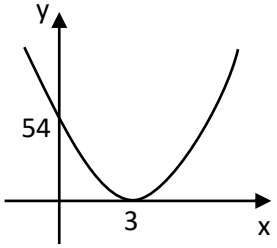
$5 = k(-3)(-3 + 6)$

$5 = -9k$

$-\frac{5}{9} = k$

$Therefore$

$y = -\frac{5}{9} x(x + 6).$



$y = k(x - 3)^2$

$When \ x = 0, \ y = 54.$

$54 = k(0 - 3)^2$

$54 = 9k$

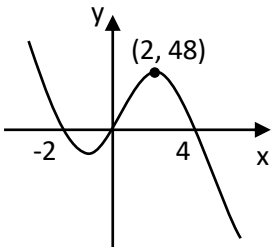
$6 = k$

$Therefore$

$y = 6(x - 3)^2.$

2.3 (b) I can identify the graph of a polynomial function.

A similar process can be used to identify the graph of a **polynomial function.**



$y = kx(x+2)(x-4)$

$When\ x = 2,\ y = 48.$

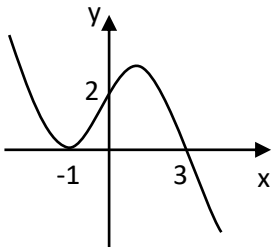
$48 = k(2)(2+2)(2-4)$

$48 = -16k$

$-3 = k$

$Therefore$

$y = -3x(x+2)(x-4).$



$y = k(x-3)(x+1)^2$

$When\ x = 0,\ y = 2.$

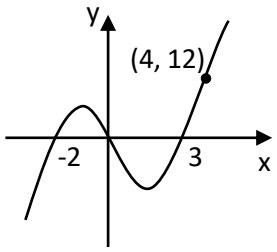
$2 = k(0-3)(0+1)^2$

$2 = -3k$

$-\frac{2}{3} = k$

$Therefore$

$y = -\frac{2}{3}(x-3)(x+1)^2.$



$y = kx(x+2)(x-3)$

$When\ x = 4,\ y = 12.$

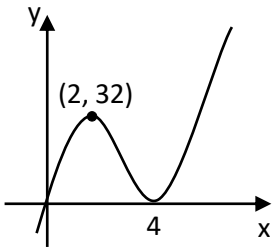
$12 = k(4)(4+2)(4-3)$

$12 = 24k$

$\frac{1}{2} = k$

$Therefore$

$y = \frac{1}{2}x(x+2)(x-3).$



$y = kx(x-4)^2$

$When\ x = 2,\ y = 32.$

$32 = k(2)(2-4)^2$

$32 = 8k$

$4 = k$

$Therefore$

$y = 4x(x-4)^2.$

2.4 Synthetic Division

2.4 (a) I can use synthetic division to factorise a polynomial.

Synthetic division allows you to **factorise polynomials** or calculate a **remainder**.

The **factor theorem** states that if $(x-a)$ is a factor of a polynomial $f(x)$ then $f(a) = 0$ and vice versa.

Note: If $(x-a)$ is a factor of a polynomial then a must be a factor of the last term.

Find the remainder when you divide $x^3 - 2x^2 + x - 5$ by $x - 2$.

2	1	−2	1	−5
		2	0	2
	1	0	1	−3

$Remainder\ is -3.$

Find the remainder when you divide $3x^3 - 9x^2 - 5x + 9$ by $x - 4$.

4	3	−9	−5	9
		12	12	28
	3	3	7	37

$Remainder\ is\ 37.$

Show that $(x-3)$ is a factor of the polynomial $x^3 - 4x^2 + x + 6$ and hence factorise it fully.

3	1	−4	1	6
		3	−3	−6
	1	−1	−2	0

$So\ (x-3)\ is\ a\ factor.$

$(x-3)(x^2 - x - 2)$

$= (x-3)(x+1)(x-2)$

Solve the equation $2x^3 + 5x^2 - 23x + 10 = 0$ using synthetic division.

2	2	5	−23	10
		4	18	−10
	2	9	−5	0

$So\ (x-2)\ is\ a\ factor.$

$(x-2)(2x^2 + 9x - 5) = 0$

$(x-2)(2x-1)(x+5) = 0$

$x = 2, \frac{1}{2}, -5$

2.4 (b) I can use synthetic division to calculate missing coefficients.

It is possible to use **synthetic division** to identify **missing coefficients**.

Find the value of k if the remainder on dividing $x^3 + 2x^2 + kx - 9$ by $(x - 2)$ is 15.

2	1	2	k	-9
		2	8	$2k + 16$
	1	4	$k + 8$	$2k + 7 = 15$

$$2k = 8$$
$$k = 4$$

Given that $(x - 1)$ is a factor of $x^3 - 5x^2 + ax - 9$, find the value of a.

1	1	-5	a	-9
		1	-4	$a - 4$
	1	-4	$a - 4$	$a - 13 = 0$

$$a = 13$$

Given that $(x - 3)$ is a factor of $x^3 - x^2 + px + 24$, find the value of p.

3	1	-1	p	24
		3	6	$3p + 18$
	1	2	$p + 6$	$3p + 42 = 0$

$$3p = -42$$
$$p = -14$$

When $f(x) = px^3 + qx^2 - 17x + 4q$ is divided by $x - 2$, the remainder is 6, and $x - 1$ is a factor of $f(x)$.

Find the values of p and q.

2	p	q	-17	$4q$
		$2p$	$4p + 2q$	$8p + 4q - 34$
	p	$2p + q$	$4p + 2q - 17$	$8p + 8q - 34 = 6$

$$8p + 8q = 40$$
$$p + q = 5$$

1	p	q	-17	$4q$
		p	$p + q$	$p + q - 17$
	p	$p + q$	$p + q - 17$	$p + 5q - 17 = 0$

$$p + 5q = 17$$

$$p + q = 5$$
$$p + 5q = 17$$
$$4q = 12$$
$$q = 3$$
$$q = 3 \Rightarrow p + 3 = 5$$
$$p = 2$$

3. Functions & Graphs

3.1 Sets & Functions

3.1 (a) I can identify the domain of a function.

The **domain** of a function is the set of values you can **put into the function** (the x-values).

Normally the domain will be the set of real numbers, however there are two types of restrictions to look out for:

1. Dividing by **zero**.
2. Finding the **square root** of a **negative number**.

Function $f(x) =$	$\sqrt{3x-15}$	$\sqrt{2+3x}$	$\sqrt{7-5x}$	$\frac{11}{x^2+4x}$	$\frac{9}{x^2-x-6}$	$\frac{7x-2}{x^2-9}$
Domain	$3x-15 \geq 0$ $3x \geq 15$ $x \geq 5$	$2+3x \geq 0$ $3x \geq -2$ $x \geq -\frac{2}{3}$	$7-5x \geq 0$ $7 \geq 5x$ $\frac{7}{5} \geq x$	$\frac{11}{x(x+4)}$ $x \neq 0, -4$	$\frac{9}{(x+2)(x-3)}$ $x \neq -2, 3$	$\frac{7x-2}{(x+3)(x-3)}$ $x \neq -3, 3$

3.1 (b) I can identify the range of a function.

The **range** of a function is the set of values **coming out of the function** (the y-values).

When finding the range of a function there are two common properties you will come need to use:

1. The **sine** and **cosine** functions range between -1 and 1.
2. A **square number** is always greater than or equal to zero.

Function	$f(x) = 5x^2 + 18$	$g(x) = 2\sin(x-9)^o$	$h(x) = \sin^2 5x^o$	$i(x) = 2\cos 3x^o + 1$
Range	$x^2 \geq 0$ $5x^2 \geq 0$ $5x^2 + 18 \geq 18$	$-1 \leq \sin(x-9)^o \leq 1$ $-2 \leq 2\sin(x-9)^o \leq 2$	$-1 \leq \sin 5x^o \leq 1$ $0 \leq \sin^2 5x^o \leq 1$	$-2 \leq \cos 3x^o \leq 2$ $3 \leq 2\cos 3x^o \leq 7$ $4 \leq 2\cos 3x^o + 1 \leq 8$

3.1 (c) I can find the formula for the composition of two functions.

A **composition** of functions occurs when you apply one function to the result of another.

If $f(x) = 3x + 2$ and $g(x) = 2 - x$, find $f(g(x))$.

$f(g(x)) = f(2-x)$
$= 3(2-x) + 2$
$= 8 - 3x$

If $a(x) = x^2 + 7$ and $b(x) = 3x - 1$, find $a(b(x))$.

$a(b(x)) = a(3x-1)$
$= (3x-1)^2 + 7$
$= 9x^2 - 6x + 8$

Given that $h(x) = x^2 + 1$ find an expression for $h(h(x))$.

$h(h(x)) = h(x^2+1)$
$= (x^2+1)^2 + 1$
$= x^4 + 2x^2 + 2$

If $f(x) = 9x + 2$ and $g(x) = 2x^2$ what is the value of $f(g(2))$?

$f(g(x)) = f(2x^2)$
$= 9(2x^2) + 2$
$= 18x^2 + 2$
$f(g(2)) = 18(2)^2 + 2$
$= 74$

3.1 (d) I can find the formula for the inverse of a given function.

The **inverse** of a function **reverses** the original function. This leads to the following properties:

- If $f(g(x)) = x$, then $f(x)$ and $g(x)$ are **inverse functions**.
- $f(f^{-1}(x)) = x$ by definition.
- The **range** of $f(x)$ is the **domain** of $f^{-1}(x)$ and the **domain** of $f(x)$ is the **range** of $f^{-1}(x)$.

To calculate the **inverse** of a function:

1. Replace $f(x)$ with y.
2. Solve for x.
3. Replace x with $f^{-1}(x)$ and y with x.

Given $f(x) = 9 - 4x$, find $f^{-1}(x)$.

Let $f(x) = y$ hence

$y = 9 - 4x$

$4x = 9 - y$

$x = \frac{9 - y}{4}$

So $f^{-1}(x) = \frac{9 - x}{4}$.

Write down an expression for $f(f^{-1}(x))$.

$f(f^{-1}(x)) = x$

Given $g(x) = 7x + 1$, find $g^{-1}(x)$.

Let $g(x) = y$ hence

$y = 7x + 1$

$y - 1 = 7x$

$\frac{y - 1}{7} = x$

So $g^{-1}(x) = \frac{x - 1}{7}$.

If $g(2) = 15$, write down the value of $g^{-1}(15)$.

$g^{-1}(15) = 2$

Given $h(x) = x^2 - 9$, find $h^{-1}(x)$.

Let $h(x) = y$ hence

$y = x^2 - 9$

$y + 9 = x^2$

$\sqrt{y + 9} = x$

So $h^{-1}(x) = \sqrt{x + 9}$.

Write down an expression for $h(h^{-1}(x))$.

$h(h^{-1}(x)) = x$

Given $i(x) = \sqrt{x} - 7$, find $i^{-1}(x)$.

Let $i(x) = y$ hence

$y = \sqrt{x} - 7$

$y + 7 = \sqrt{x}$

$(y + 7)^2 = x$

So $i^{-1}(x) = (x + 7)^2$.

If $i(4) = -5$, write down the value of $i^{-1}(-5)$.

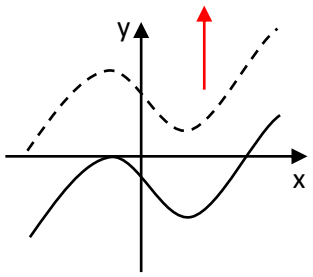
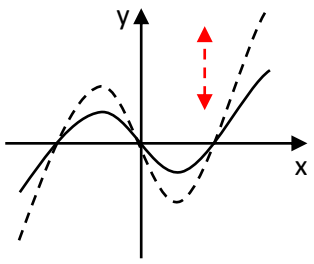
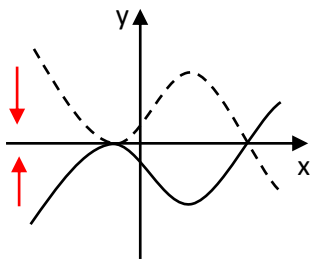
$i^{-1}(-5) = 4$

3.2 Transformations

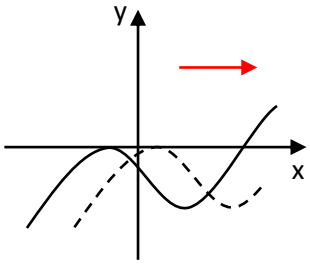
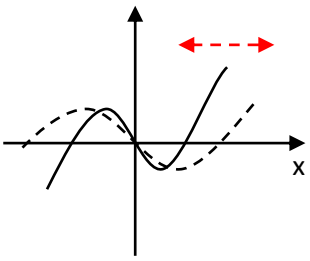
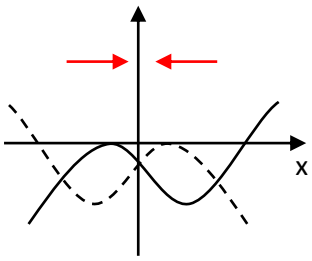
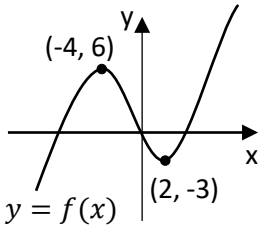
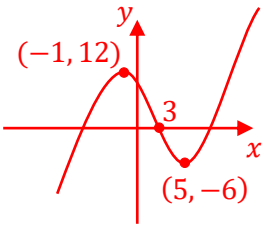
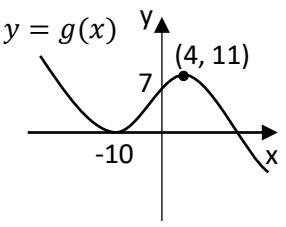
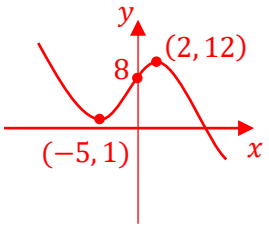
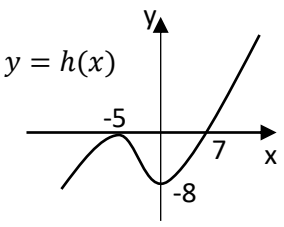
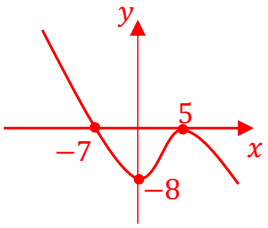
3.2 I can identify and sketch the graphs of transformations.

A function $y = f(x)$, may be **transformed** either by **translation**, **resizing**, or **reflection**.

If a function has the transformation **outside the bracket** this affects only the **y-coordinates**.

	Translation	**Resizing**	**Reflection**
Vertical (Affecting only the y-coordinates)	 $y = f(x) + a$ **translates** a units up.	 $y = af(x)$ **stretches** by a scale factor of a.	 $y = -f(x)$ **reflects** in the x-axis.

If a function has the transformation **inside the bracket** this affects only the **x-coordinates**.

	Translation	Resizing	Reflection
Horizontal (Affecting only the x-coordinates)	 $y = f(x - a)$ **translates** a units right.	 $y = f(x/a)$ **stretches** by a scale factor of a.	 $y = f(-x)$ **reflects** in the y-axis.
Sketch $y = 2f(x - 3)$.  	Sketch $y = g(2x) + 1$.  	Sketch $y = h(-x)$.  	The only stationary point of $y = f(x)$ is (a, b). What is the only stationary point of $y = -f(3x)$? $f(x)$ *has T.P.* $(a,\ b)$. $f(3x)$ *has T.P.* $\left(\frac{1}{3}a, b\right)$. $-f(3x)$ *has T.P.* $\left(\frac{1}{3}a, -b\right)$.

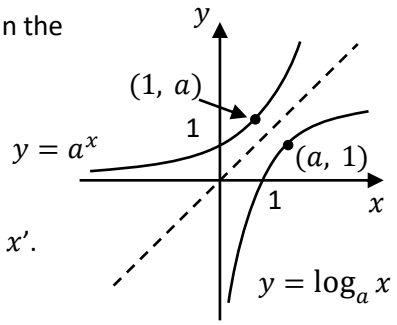
3.2 (b) I can identify and sketch the inverse of a graph.

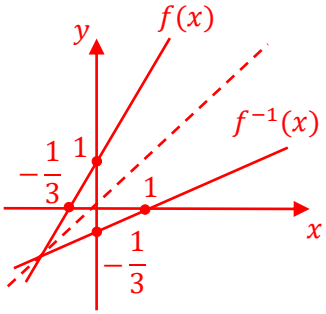
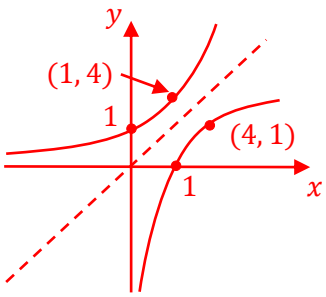
The graph of an **inverse** function can be obtained by **reflecting** the original graph in the line $y = x$, this switches the x and y values.

The **inverse** of an **exponential function** is the **logarithmic function:**

- If $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.
- If $y = a^x$ then $\log_a y = x$.
- $y = \log_a x$ means 'y is the number of times you multiply a by itself to get x'.

Note: The corresponding point for an inverse function is known as its **image**.



Sketch on the same graph $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = 3x + 1$. 	Sketch on the same graph $y = h(x)$ and $y = h^{-1}(x)$ where $h(x) = 4^x$. 	What is the inverse of $f(x) = \log_5 x$? $f^{-1}(x) = 5^x$ What is the inverse of $g(x) = 3^x$? $g^{-1}(x) = \log_3 x$ Find the image of P(2, 9). $P'(9, 2)$	$y = 3m^x$ passes through the point (5, 96). Calculate the value of m. $At\ (5, 96)\ x = 5,\ y = 96.$ $96 = 3m^5$ $32 = m^5$ $2 = m$

4. Differentiation

4.1 Basic Differentiation

4.1 (a) I can find the derivative of a basic function.

When you **differentiate** a function you get its **derivative**. You need to be able to work with two forms of notation:

Lagrange's notation	Term: $f(x)$	Derivative: $f'(x)$
Leibniz's notation	Term: y	Derivative: $\frac{dy}{dx}$

To **differentiate** a function**:**

1. Multiply by the power.
2. Decrease the power by 1.

$$f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$$

Function	$f(x) = x^{-5} + 9$	$a(t) = 2t^3 + 3t$	$y = 2x^7 - 3x^{-2} + 7$	$s = 4t^4 - 8t^2 + 2$
Derivative	$f'(x) = -5x^{-6}$	$a'(t) = 6t^2 + 3$	$\frac{dy}{dx} = 14x^6 + 6x^{-3}$	$\frac{ds}{dt} = 16t^3 - 16t$

4.1 (b) I can write a function in differentiable form and differentiate it.

From National 5: You should know the following:

Index	a^0	a^1	a^{m+n}	a^{m-n}	a^{mn}	a^{-m}	$a^{\frac{m}{n}}$
Meaning	1	a	$a^m \times a^n$	$a^m \div a^n$	$(a^m)^n$	$\frac{1}{a^m}$	$\left(\sqrt[n]{a}\right)^m$

In Higher: You should be able to write a function in **differentiable form**, using the above, and then **differentiate it**.

Function	$a(t) = 3t^2 + \frac{2}{t^2}$	$f(x) = 7\sqrt[3]{x} + 1$	$y = \frac{10}{\sqrt{x}} - 2\sqrt[3]{x}$	$m = 5\sqrt{n^3} + \frac{7}{n^4}$
Derivative	$a(t) = 3t^2 + 2t^{-2}$ $a'(t) = 6t - 4t^{-3}$ $= 6t - \frac{4}{t^3}$	$f(x) = 7x^{\frac{1}{3}} + 1$ $f'(x) = \frac{7}{3}x^{-\frac{2}{3}}$ $= \frac{7}{3\sqrt[3]{x^2}}$	$y = 10x^{-\frac{1}{2}} - 2x^{\frac{1}{3}}$ $\frac{dy}{dx} = -5x^{-\frac{1}{2}} - \frac{2}{3}x^{-\frac{2}{3}}$ $= -\frac{5}{\sqrt{x}} - \frac{2}{3\sqrt[3]{x^2}}$	$m = 5n^{\frac{3}{2}} + 7n^{-4}$ $\frac{dm}{dn} = \frac{15}{2}n^{\frac{1}{2}} - 28n^{-5}$ $= \frac{15\sqrt{n}}{2} - \frac{28}{n^5}$

4.1 (c) I can find the derivative of products and quotients.

You can also find the derivatives of **products** and **quotients** as the sum of individual terms.

Function	$a = 2b(b-1)$	$y = \sqrt{x}(2x^2 - \sqrt{x})$	$s(t) = \frac{7t-1}{t^2}$	$h(x) = \frac{9x^2+4}{\sqrt{x^3}}$
Derivative	$a = 2b^2 - b$ $\frac{da}{db} = 4b - 1$	$y = 2x^{\frac{5}{2}} - x$ $\frac{dy}{dx} = 5x^{\frac{3}{2}} - 1$ $= 5\sqrt{x^3} - 1$	$s(t) = 7t^{-1} - t^{-2}$ $s'(t) = -7t^{-2} + 2t^{-3}$ $= -\frac{7}{t^2} + \frac{2}{t^3}$	$h(x) = 9x^{\frac{1}{2}} + 4x^{-\frac{3}{2}}$ $h'(x) = \frac{9}{2}x^{-\frac{1}{2}} - 6x^{-\frac{3}{2}}$ $= \frac{9}{2\sqrt{x}} - \frac{15}{2\sqrt{x^3}}$

4.1 (d) I can calculate the rate of change of a function.

The **derivative** of a function is a measure of the **rate of change** of the function.

If $A(t) = 3t^2 - t + 10$, find the rate of change of A when $t = 5$.

$A'(t) = 6t - 1$

$A'(5) = 6(5) - 1$

$= 29$

If $F(x) = x^3 - 6x$, find the rate of change of F when $x = 4$.

$F'(x) = 3x^2 - 6$

$F'(4) = 3(4)^2 - 6$

$= 42$

If $p = 2r^4 - 10r$, find the rate of change of p when $r = 1$.

$\frac{dp}{dr} = 8r^3 - 10$

$\frac{dp}{dr}_{r=3} = 8(1)^2 - 10$

$= -2$

If $w = 7k^3 - 20k$, find the rate of change of w when $k = 10$.

$\frac{dw}{dk} = 21k^2 - 20$

$\frac{dp}{dr}_{k=10} = 21(10)^2 - 20$

$= 2\,080$

4.2 Tangents

4.2 (a) I can use differentiation to calculate the gradient of a curve.

The **derivative** of a function is a measure of the **gradient** of its graph.

Find the gradient of the tangent to $y = -4x^2 + 7$ at the point where $x = 5$.

$\frac{dy}{dx} = -8x$

$m = \frac{dy}{dx}_{x=5}$

$= -8(5)$

$= -40$

Find the gradient of the tangent to $y = 2x^3 + x^2$ at the point where $x = 3$?

$\frac{dy}{dx} = 6x^2 + 2x$

$m = \frac{dy}{dx}_{x=3}$

$= 6(3)^2 + 2(3)$

$= 60$

For $y = 3x^3 - 40x$ find the gradient of the tangent at (2, -4).

$\frac{dy}{dx} = 9x^2 - 40$

$m = \frac{dy}{dx}_{x=2}$

$= 9(2)^2 - 40$

$= -4$

Find x for which the gradient of the tangent of $y = x^3 - 9x^2$ is -27.

$\frac{dy}{dx} = 3x^2 - 18x$

$m = \frac{dy}{dx}$

$-27 = 3x^2 - 18x$

$0 = 3x^2 - 18x + 27$

$0 = 3(x^2 - 6x + 9)$

$0 = 3(x - 3)(x - 3)$

$x = 3$

4.2 (b) I can use differentiation to find the equation of a tangent.

After finding the gradient of a **tangent** to a curve you can find its equation using $y - b = m(x - a)$.

Find the tangent to $y = 4x^3 - 7x$ at the point where $x = 1$.

$x = 1 \Rightarrow y = 4(1)^3 - 7(1) = -3$

$\frac{dy}{dx} = 12x^2 - 7$

$m = \frac{dy}{dx}_{x=1}$

$= 12(1)^2 - 7 = 5$

$y - -3 = 5(x - 1)$

$y + 3 = 5x - 5$

$y = 5x - 8$

Find the tangent to $y = 5x^2 - 8x$ at the point where $x = 2$.

$x = 2 \Rightarrow y = 5(2)^2 - 8(2) = 4$

$\frac{dy}{dx} = 10x - 8$

$m = \frac{dy}{dx}_{x=2}$

$= 10(2) - 8 = 12$

$y - 4 = 12(x - 2)$

$y - 4 = 12x - 24$

$y = 12x - 20$

Find the tangent to $y = 9x - x^2$ at the point where $x = 2$.

$x = 2 \Rightarrow y = 9(2) - 2^2 = 14$

$\frac{dy}{dx} = 9 - 2x$

$m = \frac{dy}{dx}_{x=2}$

$= 9 - 2(2) = 5$

$y - 14 = 5(x - 2)$

$y - 14 = 5x - 10$

$y = 5x + 4$

4.3 Derivatives & Graphs

4.3 (a) I can identify when functions increase, decrease or are stationary.

Problems involving **increasing** and **decreasing** functions can be solved by differentiation.

- A function $f(x)$ is **increasing** if the gradient of the curve is **positive**, i.e. $f'(x) > 0$.
- A function $f(x)$ is **decreasing** if the gradient of the curve is **negative**, i.e. $f'(x) < 0$.
- A function $f(x)$ is **stationary** if the gradient of the curve is **zero**, i.e. $f'(x) = 0$.

Is $f(x) = x^3 - 12x + 3$ increasing or decreasing when $x = 1$?

$f'(x) = 3x^2 - 12$

$f'(1) = 3(1) - 12$

$= -9$

As $f'(1) < 0$, f is decreasing when $x = 1$.

Is $g(x) = 3x^2 + 5x$ increasing or decreasing when $x = 9$?

$g'(x) = 6x + 5$

$g'(10) = 6(9) + 5$

$= 59$

As $g'(1) > 0$, g is increasing when $x = 9$.

Explain why the function $f(x) = x^3 + 3x^2 + 3x$ never decreases.

$f'(x) = 3x^2 + 6x + 3$

$= 3(x^2 + 2x + 1)$

$= 3(x + 1)^2$

≥ 0 *as* $(x + 1)^2 \geq 0$

As $f'(x) \geq 0, f(x)$ never decreases.

Explain why the function $h(x) = 1/3\, x^3 - x^2 + 7x$ is always increasing.

$h'(x) = x^2 - 2x + 7$

$= (x - 1)^2 - 1 + 7$

$= (x - 1)^2 + 6$

≥ 6 *as* $(x - 1)^2 \geq 0$

As $h'(x) > 0, h(x)$ is always increasing.

4.3 (b) I can identify stationary points and their nature.

At **stationary points** a function is **neither increasing nor decreasing**, it can be one of four options:



Maximum turning point



Minimum turning point



Rising point of inflexion



Falling point of inflexion

You should be able to determine both the **coordinates** of **stationary points** and their **nature:**

1. Differentiate the function.
2. Set this derivative equal to zero and solve for x.
3. Determine their type using a '**nature table**'.
4. Find the respective y-coordinates.

You can use this process to sketch the graph of a function – remember at the y-axis x = 0 and at the x-axis y = 0.

Find the stationary points of $y = x^2 - 4x + 5$.

$\frac{dy}{dx} = 2x - 4$

$2x - 4 = 0$ *for st.pts*

$2(x - 2) = 0$

$x = 2$

x	1	2	3
$\frac{dy}{dx}$	−	0	+
Slope	\	‾	/

Find the stationary points of $y = x^3 + 3x^2$.

$\frac{dy}{dx} = 3x^2 + 6x$

$3x^2 + 6x = 0$ *for st.pts*

$3x(x + 2) = 0$

$x = 0, -2$

x	−3	−2	−1	0	1
$\frac{dy}{dx}$	+	0	−	0	+
Slope	/	‾	\	‾	/

Sketch the curve with equation $y = x(x - 9)^2$.

$y = x^3 - 18x^2 + 81x$

$\frac{dy}{dx} = 3x^2 - 36x + 81$

$3x^2 - 36x + 81 = 0$ *for st.pts*

$3(x^2 - 12x + 27) = 0$

$3(x - 9)(x - 3) = 0$

$x = 3, 9$

x	2	3	4	9	10
$\frac{dy}{dx}$	+	0	−	0	+
Slope	/	‾	\	‾	/

$x = 2 \Rightarrow y = 2^2 - 4(2) + 5$

$= 1$

$Min\ T.P.\ at\ (2, 1).$

$x = 0 \Rightarrow y = (0)^3 + 3(0)^2$

$= 0$

$x = -2 \Rightarrow y = (-2)^3 + 3(-2)^2$

$= 4$

$Max\ T.P.\ at\ (-2, 4).$

$Min\ T.P.\ at\ (0, 0).$

$x = 3 \Rightarrow y = 3(3 - 9)^2$

$= 108$

$x = 9 \Rightarrow y = 9(9 - 9)^2$

$= 0$

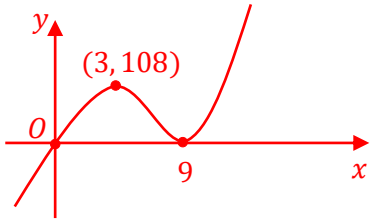
$Max\ T.P.\ at\ (3, 108).$

$Min\ T.P.\ at\ (9, 0).$

$On\ the\ x\text{-}axis\ y = 0.$

$0 = x(x - 9)^2$

$Coordinates\ (0,\ 0)\ \&\ (9, 0).$

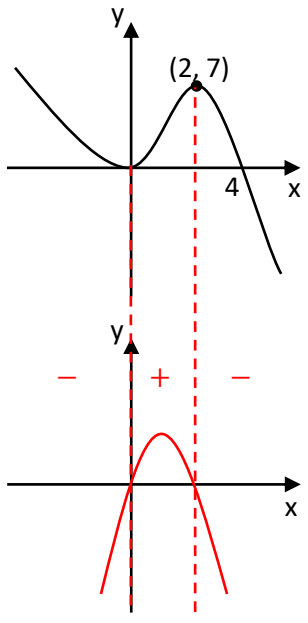


4.3 (c) I can sketch the graph of a derived function.

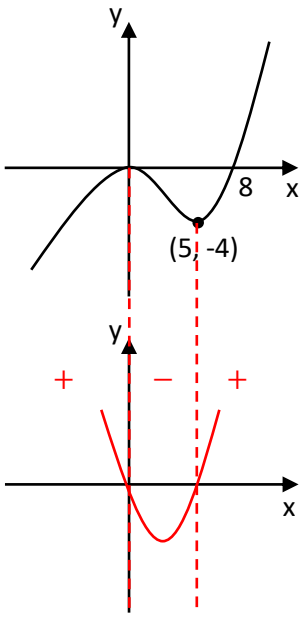
You may be given the graph of $f(x)$ and asked to sketch the graph of the derivative $f'(x)$, to do so:

1. Transfer the **stationary points** as zeros onto the new graph
2. Where $y = f(x)$ is **increasing**, the graph of $y = f'(x)$ is **positive** (above the x-axis).
3. Where $y = f(x)$ is **decreasing**, the graph of $y = f'(x)$ is **negative** (below the x-axis).
4. Connect these zeros by smooth curves.

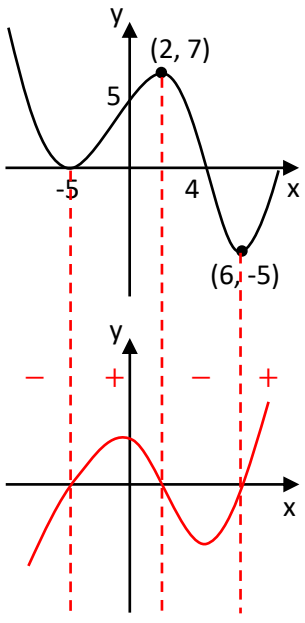
Sketch $f'(x)$.



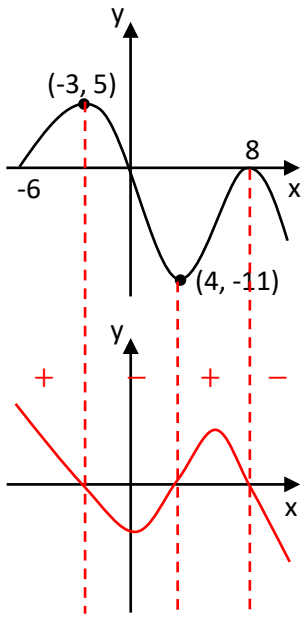
Sketch $f'(x)$.



Sketch $f'(x)$.



Sketch $f'(x)$.



4.4 Optimisation

4.4 (a) I can optimise a function in a closed interval.

The **maximum** and **minimum** values of a function in a **closed interval** occur either at **turning points** or **endpoints**.

Find the maximum and minimum values of $f(x) = x^3 - 3x^2 + 2$ in the closed interval $-3 \leq x \leq 3$.

$f'(x) = 3x^2 - 6x$

$3x^2 - 6x = 0 \; for \; st.pts$

$3x(x-2) = 0$

$x = 0,\ 2$

$f(-3) = (-3)^3 - 3(-3)^2 + 2$

$= -52$

$f(0) = 0^3 - 3(0)^2 + 2$

$= 2$

$f(2) = 2^3 - 3(2)^2 + 2$

$= -2$

$f(3) = 3^3 - 3(3)^2 + 2$

$= 2$

$Max \; value \; is \; 2 \; where \; x = 0, 3.$

$Min \; value \; is \; -52 \; where \; x = -3.$

Find the maximum and minimum values of $g(x) = 2x^3 - 12x^2$ in the closed interval $-1 \leq x \leq 6$.

$g'(x) = 6x^2 - 24x$

$6x^2 - 24x = 0 \; for \; st.pts$

$6x(x-4) = 0$

$x = 0,\ 4$

$g(-1) = 2(-1)^3 - 12(-1)^2$

$= -14$

$g(0) = 2(0)^3 - 12(0)^2$

$= 0$

$g(4) = 2(4)^3 - 12(4)^2$

$= -64$

$g(6) = 2(6)^3 - 12(6)^2$

$= 0$

$Max \; value \; is \; 0 \; where \; x = 0, 6.$

$Min \; value \; is \; -64 \; where \; x = 4.$

Find the maximum and minimum values of $h(x) = 3x^3 - 36x$ in the closed interval $-3 \leq x \leq 4$.

$h'(x) = 9x^2 - 36$

$9x^2 - 36 = 0 \; for \; st.pts$

$9(x^2 - 4) = 0$

$9(x+2)(x-2) = 0$

$x = -2,\ 2$

$h(-3) = 3(-3)^3 - 36(-3)$

$= 27$

$h(-2) = 3(-2)^3 - 36(-2)$

$= 48$

$h(2) = 3(2)^3 - 36(2)$

$= -48$

$h(3) = 3(3)^3 - 36(3)$

$= -27$

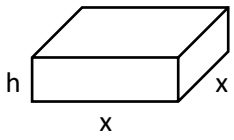
$Max \; value \; is \; 48 \; where \; x = -2.$

$Min \; value \; is \; -48 \; where \; x = 2.$

4.4 (b) I can use optimisation to solve practical problems.

Optimisation is the process of finding the greatest or smallest possible value of a problem.

The volume of the square-based glass display case shown is 500 cm^2.



The length of the base is x cm. The base is not made of glass.

Show that the area of glass in the case is:

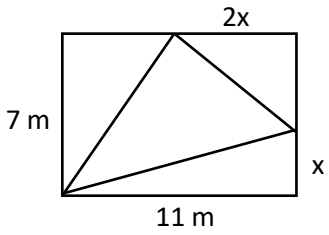
$A(x) = x^2 + \frac{2000}{x}.$

$V = x \times x \times h$

$500 = hx^2$

$\frac{500}{x^2} = h$

A rectangle ABCD measures 7 units by 11 units, as shown.

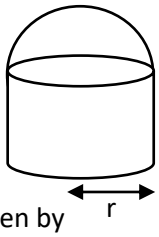


Show that the area, of the triangle is given by

$A(x) = x^2 - \frac{11}{2}x + \frac{77}{2}.$

$A = 11 \times 7 - \frac{1}{2} \times 11 \times x - \frac{1}{2} \times (7-x) \times 2x - \frac{1}{2} \times 7 \times (11 - 2x)$

A beaker has a cylinder with radius r cm and height h cm. The volume of the cylinder is 400 cm^3.



Show that the surface area of plastic A(r), needed to make the beaker is given by

$A(r) = 3\pi r^2 + \frac{800}{r}.$

Note: The curved surface area of a hemisphere is $2\pi r^2$.

$V = \pi r^2 h$

$400 = \pi r^2 h$

$\frac{400}{\pi r^2} = h$

$$A = x^2 + 4xh$$
$$= x^2 + 4x\left(\frac{500}{x^2}\right)$$
$$= x^2 + \frac{2000}{x}$$

Hence find the dimensions of the case that minimise the use of glass and calculate this minimum area.

$$A(x) = x^2 + 2000x^{-1}$$
$$A'(x) = 2x - 2000x^{-2}$$
$$2x - 2000x^{-2} = 0 \; for \; st. \; pts$$
$$2x^3 - 2000 = 0$$
$$2x^3 = 2000$$
$$x^3 = 1000$$
$$x = 10$$

x	9	10	11
$A'(x)$	$-$	0	$+$
$Slope$	\	—	/

$Min\ T.P.\ at\ x = 10.$

$Dimensions: 10 \times 10 \times 5\ (cm).$

$$A = 10^2 + \frac{2000}{10}$$
$$= 300\ cm^2$$

$$A = 77 - \frac{11x}{2} - 7x + x^2 - \frac{77}{2} + 7x$$
$$A = x^2 - \frac{11}{2}x + \frac{77}{2}$$

Hence find the value of x for which the shaded area will be the smallest.

$$A'(x) = 2x - \frac{11}{2}$$
$$2x - \frac{11}{2} = 0 \; for \; st. \; pts$$
$$2x - 11 = 0$$
$$2x = 11$$
$$x = \frac{11}{2}$$

x	5	$\frac{11}{2}$	6
$A'(x)$	$-$	0	$+$
$Slope$	\	—	/

$Min\ T.P.\ at\ x = \frac{11}{2}.$

$$S.A. = 2\pi r^2 + 2\pi rh + \pi r^2$$
$$= 3\pi r^2 + 2\pi r\left(\frac{400}{\pi r^2}\right)$$
$$= 3\pi r^2 + \frac{800}{r}$$

Find the value of r which ensures that the surface area of plastic is minimised.

$$A(r) = 3\pi r^2 + 800r^{-1}$$
$$A'(r) = 6\pi r - 800r^{-2}$$
$$6\pi r - 800r^{-2} = 0 \; for \; st. \; pts$$
$$6\pi r^3 - 800 = 0$$
$$6\pi r^3 = 800$$
$$r^3 = \frac{400}{3\pi}$$
$$r = \sqrt[3]{\frac{400}{3\pi}}$$

r	3	$\sqrt[3]{\frac{400}{3\pi}}$	4
$A'(r)$	$-$	0	$+$
$Slope$	\	—	/

$Min\ T.P.\ at\ r = \sqrt[3]{\frac{400}{3\pi}}.$

5. Trigonometric Equations

5.1 Trig Graphs

5.1 (a) I can convert between degrees and radians.

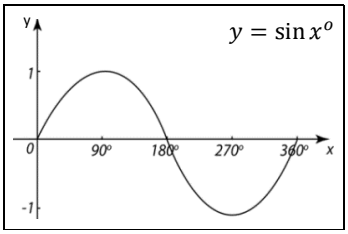
You should be able to convert between **degrees** and **radians**. The staring point is 180° = π radians.

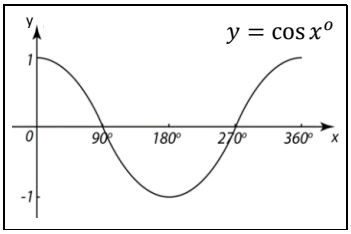
Degrees	36^o	60^o	90^o	120^o	135^o	225^o	240^o	270^o	300^o	330^o
Radians	$\frac{\pi}{5}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$

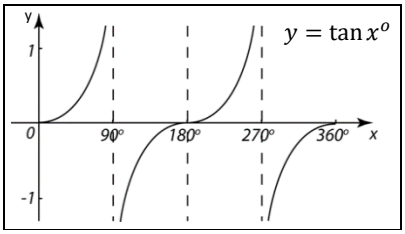
5.1 (b) I can identify exact values in both degrees and radians.

From National 5: You should know the following:

The graphs of the functions $y = \sin x^o$, $y = \cos x^o$, and $y = \tan x^o$.

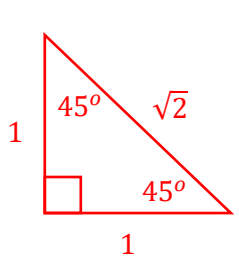


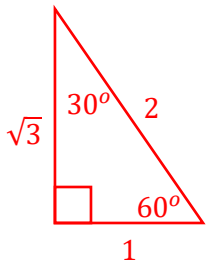




In Higher: You should know the following **exact values**, two triangles can help in this:

	30^o	45^o	60^o
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

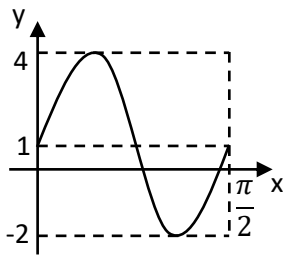




5.1 (c) I can identify the equation of a trigonometric graph.

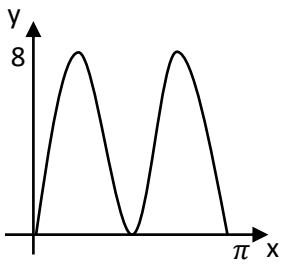
You should also be able to identity, the equation of a **trigonometric graph**.

$y = p \sin qx + r$ find p, q and r.



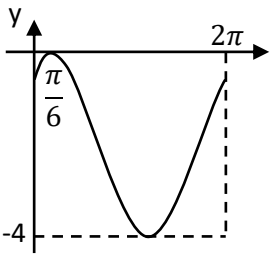
$p = 3, q = 4, r = 1$

$y = a \cos bx + c$ find a, b and c.



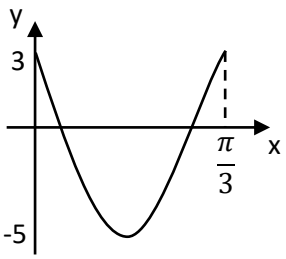
$a = -4, b = 4, c = 4$

$y = m \cos(x - n) + k$ find m, n and k.



$m = 2, n = \frac{\pi}{6}, k = -2$

$y = d \cos ex + f$ find d, e and f.



$d = 4, e = 6, f = -1$

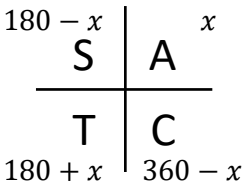
5.2 Trig Equations

5.2 (a) I can solve basic trigonometric equations.

From National 5: You should know the following:

To solve a trigonometric equation between $0 \leq x < 360$.

1. **Rearrange** the equation to get $\sin x$, $\cos x$ or $\tan x$ alone on one side.
2. Take the **inverse** using your calculator.
3. Use the **CAST** diagram to identify the **quadrants**.
4. Use these to give you your **two solutions.**



In Higher: You may be asked to solve more difficult equations:

- You may be asked to work with **degrees** or **radians**.
- You have to be careful of the period you are working with. If your answer exceeds 360° take this away.
- You can be asked to work with compound angles, $(x + 45)^o$, $(2x - 30)^o$ etc.

Solve $3\sin x^o = -1$, $0 \leq x \leq 360$.

$\sin x^o = -\frac{1}{3}$

$x^o = \sin^{-1}\left(-\frac{1}{3}\right)$

$= (180 + 26.57)^o, (360 - 26.57)^o$

$= 206.57^o, 333.43^o$

Solve $5\tan(x + 30)^o = 4$ $0 \leq x \leq 360$.

$\tan(x + 30)^o = \frac{4}{5}$

$(x + 30)^o = \tan^{-1}\left(\frac{4}{5}\right)$

$= 38.66^o, (180 + 38.66)^o$

$= 38.66^o, 218.66^o$

$x^o = 8.66^o, 188.66^o$

Solve $3 = 4 - \tan 2x$, $0 \leq x < 2\pi$.

$\tan 2x = 1$

$2x = \tan^{-1}(1)$

$= \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$= \frac{\pi}{4}, \frac{5\pi}{4}$

$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

$(period\ of\ \pi)$

Solve $2\tan 2x - 1 = 0$, $0 \leq x < 2\pi$.

$4\tan 2x = 1$

$\tan 2x = \frac{1}{2}$

$2x = \tan^{-1}\left(\frac{1}{2}\right)$

$= 0.46, \pi + 0.46$

$x = 0.23, 1.81, 3.37, 4.95$

$(period\ of\ \pi)$

5.2 (b) I can solve trigonometric equations involving factorisation.

You may also be asked to solve trigonometric equations involving **factorisation**.

Note: Some **trigonometric equations** have **NO** solutions (such as $\sin x = 3$).

Solve the equation $2\cos^2 x^o + \cos x^o - 1 = 0$, $0 \leq x \leq 360$.

$(2\cos x^o - 1)(\cos x^o + 1) = 0$

$\cos x^o = \frac{1}{2}$

$x^o = \cos^{-1}\left(\frac{1}{2}\right)$

$= 60^o, (360 - 60)^o$

$= 60^o, 300^o$

$\cos x^o = -1$

$x^o = \cos^{-1}(-1)$

$= 180^o$

Solve the equation $\sin x \cos x - 3\cos x = 0$, $0 \leq x \leq 2\pi$.

$\cos x\,(\sin x - 3) = 0$

$\cos x = 0$

$x = \cos^{-1}(0)$

$= \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x = 3$

this has no solutions

6. Vectors

6.1 Vector Calculations

6.1 (a) I can find the components of a vector joining two points.

From National 5: You should know the following:

Vectors are quantities with both a **size** and a **direction**. They can be either two or three dimensional.

- To multiply a vector $\boldsymbol{a}$ and a scalar k multiply the components of $\boldsymbol{a}$ by k.
- To add vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ simply add their components.

In Higher: Where $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$ you can find the **components** of vector $\overrightarrow{AB}$ using the rule:

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

Find $\overrightarrow{AB}$ where $A(5, 1, -3)$ and $B(7, 0, 8)$.

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 7 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 11 \end{pmatrix}$$

Find $\overrightarrow{EF}$ where $E(7, -4, -3)$ and $F(-2, 4, -7)$.

$$\overrightarrow{EF} = \boldsymbol{f} - \boldsymbol{e} = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ -7 \end{pmatrix} = \begin{pmatrix} 9 \\ -8 \\ 4 \end{pmatrix}$$

Find $\overrightarrow{MN}$ where $M(2, 0, 3)$ and $N(-3, 2, 1)$.

$$\overrightarrow{MN} = \boldsymbol{n} - \boldsymbol{m} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix}$$

Find $\overrightarrow{PQ}$ where $P(-3, 5, 4)$ and $Q(2, -9, -1)$.

$$\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 2 \\ -9 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -14 \\ -5 \end{pmatrix}$$

6.1 (b) I can calculate distances in 3 dimensions and magnitudes.

From National 5: You should know the following:

The **magnitude** of vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is its length this is found using: $|\boldsymbol{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

In Higher: To calculate the **distance** between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) use:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Find $|\boldsymbol{u}|$ if $\boldsymbol{u} = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$.

$$|\boldsymbol{u}| = \sqrt{(-4)^2 + 0^2 + 3^2} = 5$$

Find $|\boldsymbol{a}|$ if $\boldsymbol{a} = \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$.

$$|\boldsymbol{a}| = \sqrt{3^2 + (-5)^2 + 7^2} = \sqrt{83}$$

How far apart are (2, 4, 3) and (10, 0, -5)?

$$d = \sqrt{8^2 + 4^2 + 8^2} = 12$$

How far apart are (-3, 0, 2) and (5, 5, -7)?

$$d = \sqrt{8^2 + 5^2 + 9^2} = \sqrt{170}$$

6.1 (c) I can identify vector pathways.

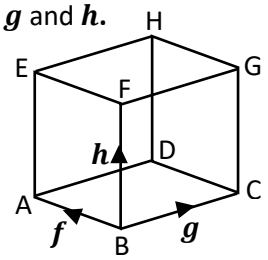
From National 5: You should know the following:

You can express a vector in a diagram as a combination of other, known, vectors: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

If you move **backwards** along a vector, **take the vector away**: $\overrightarrow{AB} = -\overrightarrow{BA}$.

In Higher: You can be given vector pathways questions involving **three dimensions**.

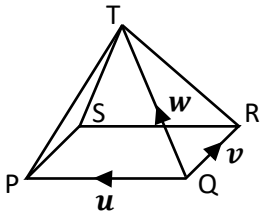
For the cuboid below, express $\overrightarrow{HB}$ in terms of $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$.



$\overrightarrow{HB} = \overrightarrow{HG} + \overrightarrow{GF} + \overrightarrow{FB}$

$= -\boldsymbol{f} - \boldsymbol{g} - \boldsymbol{h}$

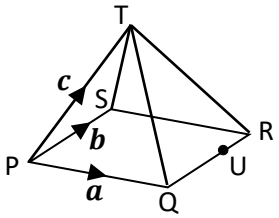
For the square based pyramid, express $\overrightarrow{TS}$ in terms of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$.



$\overrightarrow{TS} = \overrightarrow{TQ} + \overrightarrow{QR} + \overrightarrow{RS}$

$= -\boldsymbol{w} + \boldsymbol{v} + \boldsymbol{u}$

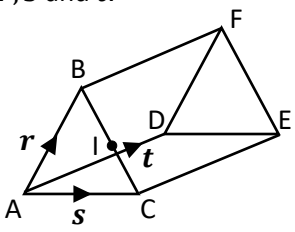
U divides QR in the ratio 3:1. Express $\overrightarrow{TU}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.



$\overrightarrow{TU} = \overrightarrow{TP} + \overrightarrow{PQ} + \frac{3}{4}\overrightarrow{QR}$

$= -\boldsymbol{c} + \boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

I is the midpoint of BC. Express $\overrightarrow{FI}$ in terms of $\boldsymbol{r}, \boldsymbol{s}$ and $\boldsymbol{t}$.



$\overrightarrow{FI} = \overrightarrow{FB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$

$= -\boldsymbol{t} - \frac{1}{2}\boldsymbol{r} + \frac{1}{2}\boldsymbol{s}$

6.1 (d) I can identify and find unit vectors.

A **unit vector** is any vector which has a length of 1 unit. Three important unit vectors are $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. These are parallel to the x, y and z-axes respectively.

Therefore $\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. So $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} = \begin{pmatrix}a\\b\\c\end{pmatrix}$. To find the components of a **unit vector**:

1. Find the **magnitude** of the given vector.
2. **Divide** the components by the magnitude.

If $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} - 3\boldsymbol{k}$ then find $|\boldsymbol{a} + \boldsymbol{b}|$.

$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \begin{pmatrix}5\\0\\-3\end{pmatrix}$

$= \begin{pmatrix}7\\-1\\1\end{pmatrix}$

$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{7^2 + 1^2 + 1^2}$

$= \sqrt{51}$

Find the unit vector parallel to $\boldsymbol{m} = \begin{pmatrix}6\\0\\-8\end{pmatrix}$.

$|\boldsymbol{m}| = \sqrt{6^2 + 0^2 + (-8)^2}$

$= 10$

$Unit\ vector = \begin{pmatrix}\frac{3}{5}\\0\\-\frac{4}{5}\end{pmatrix}$

Find the unit vector parallel to $\boldsymbol{n} = \begin{pmatrix}4\\-2\\3\end{pmatrix}$.

$|\boldsymbol{n}| = \sqrt{4^2 + (-2)^2 + 3^2}$

$= \sqrt{29}$

$Unit\ vector = \begin{pmatrix}\frac{4}{\sqrt{29}}\\-\frac{2}{\sqrt{29}}\\\frac{3}{\sqrt{29}}\end{pmatrix}$

Find a unit vector parallel to $\boldsymbol{u} = 5\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$.

$|\boldsymbol{u}| = \sqrt{5^2 + (-1)^2 + 7^2}$

$= \sqrt{75}$

$Unit\ vector = \begin{pmatrix}\frac{5}{\sqrt{75}}\\-\frac{1}{\sqrt{75}}\\\frac{7}{\sqrt{75}}\end{pmatrix}$

6.2 Vectors & Ratio

6.2 (a) I can identify and work with collinear points.

If $\overrightarrow{AB} = k\overrightarrow{CD}$ where k is a scalar, then AB is **parallel** to CD.

If $\overrightarrow{AB} = k\overrightarrow{BC}$ where k is a scalar and they also have a point B in **common,** A, B and C are **collinear**.

Are A(3, 2, 4), B(6, 3, 6) and C(12, 5, 10) collinear?

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= \begin{pmatrix}6\\3\\6\end{pmatrix} - \begin{pmatrix}3\\2\\4\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}$

$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

$= \begin{pmatrix}12\\5\\10\end{pmatrix} - \begin{pmatrix}6\\3\\6\end{pmatrix} = \begin{pmatrix}6\\2\\4\end{pmatrix}$

$2\overrightarrow{AB} = \overrightarrow{BC}$ *therefore* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$ *parallel and have point B in common therefore A, B and C are collinear.*

Are P(2, -4, 5), Q(3, 0, 8) and R(8, 6, 14) collinear?

$\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$

$= \begin{pmatrix}3\\0\\8\end{pmatrix} - \begin{pmatrix}2\\-4\\5\end{pmatrix} = \begin{pmatrix}1\\4\\3\end{pmatrix}$

$\overrightarrow{QR} = \boldsymbol{c} - \boldsymbol{b}$

$= \begin{pmatrix}8\\6\\14\end{pmatrix} - \begin{pmatrix}3\\0\\8\end{pmatrix} = \begin{pmatrix}3\\6\\6\end{pmatrix}$

$\overrightarrow{PQ}$ *and* $\overrightarrow{QR}$ *are not parallel and therefore P, Q and R are not collinear.*

Given $R(3, 0, 6)$ and $\overrightarrow{RT} = 2\overrightarrow{RS}$ where $\overrightarrow{RS} = \begin{pmatrix}4\\1\\2\end{pmatrix}$ find the coordinates of T.

$\overrightarrow{RT} = 2\overrightarrow{RS}$

$= 2\begin{pmatrix}4\\1\\2\end{pmatrix} = \begin{pmatrix}8\\2\\4\end{pmatrix}$

$\boldsymbol{t} = \boldsymbol{r} + \overrightarrow{RT}$

$= \begin{pmatrix}3\\0\\6\end{pmatrix} + \begin{pmatrix}8\\2\\4\end{pmatrix} = \begin{pmatrix}11\\2\\10\end{pmatrix}$

So $T(11,\ 2,\ 10)$.

Given $A(2, 5, 4)$ and $\overrightarrow{AC} = 3\overrightarrow{AB}$ where $\overrightarrow{AB} = \begin{pmatrix}2\\0\\3\end{pmatrix}$ find the coordinates of C.

$\overrightarrow{AC} = 3\overrightarrow{AB}$

$= 3\begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}6\\0\\9\end{pmatrix}$

$\boldsymbol{c} = \boldsymbol{a} + \overrightarrow{AC}$

$= \begin{pmatrix}2\\5\\4\end{pmatrix} + \begin{pmatrix}6\\0\\9\end{pmatrix} = \begin{pmatrix}8\\5\\13\end{pmatrix}$

So $C(8,\ 5,\ 13)$.

6.2 (b) I can find the ratio in which a point divides a line.

To find the **ratio** in which a **point** B divides a line AC, find and compare vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Given that $\overrightarrow{MN} = \begin{pmatrix}4\\4\\8\end{pmatrix}, \overrightarrow{NO} = \begin{pmatrix}3\\3\\6\end{pmatrix}$ find the ratio in which N divides MO.

$\overrightarrow{MN} : \overrightarrow{NO}$

$\begin{pmatrix}4\\4\\8\end{pmatrix} : \begin{pmatrix}3\\3\\6\end{pmatrix}$

$4 : 3$

If D(-2, 8, 3) E(8, 3, -2) and F are collinear and $\overrightarrow{EF} = \begin{pmatrix}4\\-2\\-2\end{pmatrix}$ find the ratio which E divides DF.

$\overrightarrow{DE} = \boldsymbol{e} - \boldsymbol{d}$

$= \begin{pmatrix}8\\3\\-2\end{pmatrix} - \begin{pmatrix}-2\\8\\3\end{pmatrix}$

$= \begin{pmatrix}10\\-5\\-5\end{pmatrix}$

$\overrightarrow{DE} : \overrightarrow{EF}$

$\begin{pmatrix}10\\-5\\-5\end{pmatrix} : \begin{pmatrix}4\\-2\\-2\end{pmatrix}$

$5 : 2$

Given that the coordinates A(3, 0, 7), B(7, 2, 3) and C(15, 6, -5) are collinear, calculate the ratio in which B divides AC.

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= \begin{pmatrix}7\\2\\3\end{pmatrix} - \begin{pmatrix}3\\0\\7\end{pmatrix} = \begin{pmatrix}4\\2\\-4\end{pmatrix}$

$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

$= \begin{pmatrix}15\\6\\-5\end{pmatrix} - \begin{pmatrix}7\\2\\3\end{pmatrix} = \begin{pmatrix}8\\4\\-8\end{pmatrix}$

$\overrightarrow{AB} : \overrightarrow{BC}$

$\begin{pmatrix}4\\2\\-4\end{pmatrix} : \begin{pmatrix}8\\4\\-8\end{pmatrix}$

$1 : 2$

Given that the coordinates L(3, 5, 8), M(9, 14, 11) and N(13, 20, 13) are collinear, find the ratio in which M divides LN.

$\overrightarrow{LM} = \boldsymbol{m} - \boldsymbol{l}$

$= \begin{pmatrix}9\\14\\11\end{pmatrix} - \begin{pmatrix}3\\5\\8\end{pmatrix} = \begin{pmatrix}6\\9\\3\end{pmatrix}$

$\overrightarrow{MN} = \boldsymbol{n} - \boldsymbol{m}$

$= \begin{pmatrix}13\\20\\13\end{pmatrix} - \begin{pmatrix}9\\14\\11\end{pmatrix} = \begin{pmatrix}4\\6\\2\end{pmatrix}$

$\overrightarrow{LM} : \overrightarrow{MN}$

$\begin{pmatrix}6\\9\\3\end{pmatrix} : \begin{pmatrix}4\\6\\2\end{pmatrix}$

$3 : 2$

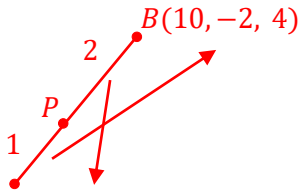
6.2 (c) I can use a ratio to identify a point on a line.

The **section formula** states that if a point P divides a line AB in the **ratio** $m : n$ then:

$$p = \frac{m\boldsymbol{b} + n\boldsymbol{a}}{m + n}$$

P lies between A(4, 4, 10) and B(10, -2, 4) in the ratio 1:2.

Find the coordinates of P.



$$\boldsymbol{p} = \frac{1}{3}\left[\begin{pmatrix}10\\-2\\4\end{pmatrix} + 2\begin{pmatrix}4\\4\\10\end{pmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix}10\\-2\\4\end{pmatrix} + \begin{pmatrix}8\\8\\20\end{pmatrix}\right]$$

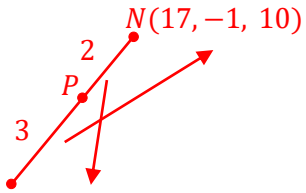
$$= \frac{1}{3}\begin{pmatrix}18\\6\\24\end{pmatrix}$$

$$= \begin{pmatrix}6\\2\\8\end{pmatrix}$$

$Therefore\ P(6,\ 2,\ 8).$

P lies between M(2, 4, 0) and N(17, -1, 7) in the ratio 3:2.

Find the coordinates of P.



$$\boldsymbol{p} = \frac{1}{5}\left[3\begin{pmatrix}17\\-1\\10\end{pmatrix} + 2\begin{pmatrix}2\\4\\0\end{pmatrix}\right]$$

$$= \frac{1}{5}\left[\begin{pmatrix}51\\-3\\30\end{pmatrix} + \begin{pmatrix}4\\8\\0\end{pmatrix}\right]$$

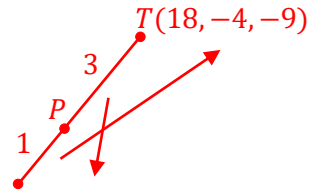
$$= \frac{1}{5}\begin{pmatrix}55\\5\\30\end{pmatrix}$$

$$= \begin{pmatrix}11\\1\\6\end{pmatrix}$$

$Therefore\ P(11,\ 1,\ 6).$

P lies between S(6, 0, 7) and T(18, -4, -9) in the ratio 3:1.

Find the coordinates of P.



$$\boldsymbol{p} = \frac{1}{4}\left[\begin{pmatrix}18\\-4\\-9\end{pmatrix} + 3\begin{pmatrix}6\\-8\\7\end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix}18\\-4\\-9\end{pmatrix} + \begin{pmatrix}18\\-24\\21\end{pmatrix}\right]$$

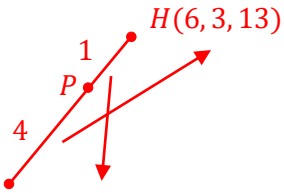
$$= \frac{1}{4}\begin{pmatrix}36\\-28\\12\end{pmatrix}$$

$$= \begin{pmatrix}9\\-7\\3\end{pmatrix}$$

$Therefore\ P(9, -7,\ 3).$

P lies between G(-4, 8, -7) and H(6, 3, 13) in the ratio 4:1.

Find the coordinates of P.



$$\boldsymbol{p} = \frac{1}{5}\left[4\begin{pmatrix}6\\3\\13\end{pmatrix} + \begin{pmatrix}-4\\8\\-7\end{pmatrix}\right]$$

$$= \frac{1}{5}\left[\begin{pmatrix}24\\12\\52\end{pmatrix} + \begin{pmatrix}-4\\8\\-7\end{pmatrix}\right]$$

$$= \frac{1}{5}\begin{pmatrix}20\\20\\45\end{pmatrix}$$

$$= \begin{pmatrix}4\\4\\9\end{pmatrix}$$

$Therefore\ P(4,\ 4,\ 9).$

6.3 Scalar Product

6.3 (a) I can calculate the scalar product of two vectors.

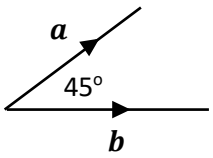
For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the **scalar** (or **dot**) **product** is defined as:

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, $0 \leq \theta \leq 180$.

Note: The vectors point away from the vertex. The scalar product is **not** a vector it is a scalar.

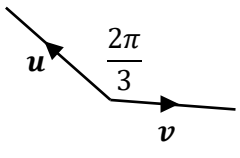
Calculate $\boldsymbol{a}.\boldsymbol{b}$ where $|\boldsymbol{a}| = 6$ and $|\boldsymbol{b}| = 5$.



$$\boldsymbol{a}.\boldsymbol{b} = 6 \times 5 \times \cos 45^o$$

$$= \frac{30}{\sqrt{2}}$$

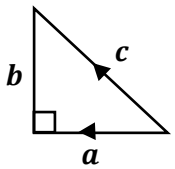
Calculate $\boldsymbol{u}.\boldsymbol{v}$ where $|\boldsymbol{u}| = 4$ and $|\boldsymbol{v}| = 7$.



$$\boldsymbol{u}.\boldsymbol{v} = 4 \times 7 \times \cos\frac{2\pi}{3}$$

$$= -14$$

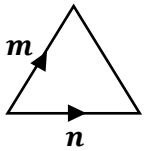
Find $\boldsymbol{a}.\boldsymbol{c}$ where $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 2$, and $|\boldsymbol{c}| = \sqrt{8}$.



$$\boldsymbol{a}.\boldsymbol{c} = 2 \times \sqrt{8} \times \cos 45^o$$

$$= 4$$

Find $\boldsymbol{m}.\boldsymbol{n}$ where the triangle is equilateral with sides 4 units long.



$$\boldsymbol{m}.\boldsymbol{n} = 4 \times 4 \times \cos 60^o$$

$$= 8$$

The **scalar product** can also be calculated if you have two vectors in **component form**:

$$\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3 \text{ where } \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Find $\boldsymbol{u}.\boldsymbol{v}$ where $\boldsymbol{u} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 4\boldsymbol{i} - 3\boldsymbol{k}$.	Find $\boldsymbol{a}.\boldsymbol{b}$ where $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.	Find $\boldsymbol{m}.\boldsymbol{n}$ where $\boldsymbol{m} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \boldsymbol{n} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.	Find $\boldsymbol{g}.\boldsymbol{h}$ where $\boldsymbol{g} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \boldsymbol{h} = \begin{pmatrix} -4 \\ 0 \\ 9 \end{pmatrix}$.
$\boldsymbol{u}.\boldsymbol{v}$ $= 3(4) + 2(0) - 1(-3)$ $= 15$	$\boldsymbol{a}.\boldsymbol{b}$ $= 4(5) + 1(-2) - 7(1)$ $= 11$	$\boldsymbol{m}.\boldsymbol{n}$ $= 2(1) - 3(0) + 6(-2)$ $= -10$	$\boldsymbol{g}.\boldsymbol{h}$ $= 5(-4) - 1(0) + 3(9)$ $= 7$

6.3 (b) I can calculate an angle using the scalar product.

You can rearrange the formula $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$ to calculate the angle between the two vectors:

$$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Calculate the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ where $\boldsymbol{u} = 4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{v} = 5\boldsymbol{i} + \boldsymbol{j}$.	Calculate the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ where $\boldsymbol{a} = 5\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.	Calculate the angle between $\boldsymbol{m}$ and $\boldsymbol{n}$ where $\boldsymbol{m} = 3\boldsymbol{i} - 7\boldsymbol{k}$ and $\boldsymbol{n} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.	Calculate the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ where $\boldsymbol{f} = 2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{g} = \boldsymbol{i} - 7\boldsymbol{j} + 2\boldsymbol{k}$.
$\lvert\boldsymbol{u}\rvert = \sqrt{4^2 + 1^2 + (-2)^2}$ $= \sqrt{21}$ $\lvert\boldsymbol{v}\rvert = \sqrt{5^2 + 1^2 + 0^2}$ $= \sqrt{26}$ $\boldsymbol{u}.\boldsymbol{v}$ $= 4(5) + 1(1) - 2(0)$ $= 21$ $\cos\theta = \frac{21}{\sqrt{21}\sqrt{26}}$ $\theta = \cos^{-1}\left(\frac{21}{\sqrt{21}\sqrt{26}}\right)$ $= 26.01^o$	$\lvert\boldsymbol{a}\rvert = \sqrt{5^2 + (-3)^2 + 1^2}$ $= \sqrt{35}$ $\lvert\boldsymbol{b}\rvert = \sqrt{1^2 + 2^2 + (-1)^2}$ $= \sqrt{6}$ $\boldsymbol{a}.\boldsymbol{b}$ $= 5(1) - 3(2) + 1(-1)$ $= -2$ $\cos\theta = \frac{-2}{\sqrt{35}\sqrt{6}}$ $\theta = \cos^{-1}\left(\frac{-2}{\sqrt{35}\sqrt{6}}\right)$ $= (180 - 82.07)^o$ $= 97.93^o$	$\lvert\boldsymbol{m}\rvert = \sqrt{3^2 + 0^2 + (-7)^2}$ $= \sqrt{58}$ $\lvert\boldsymbol{n}\rvert = \sqrt{3^2 + 1^2 + (-1)^2}$ $= \sqrt{11}$ $\boldsymbol{m}.\boldsymbol{n}$ $= 3(1) + 0(1) - 7(-1)$ $= 10$ $\cos\theta = \frac{10}{\sqrt{58}\sqrt{11}}$ $\theta = \cos^{-1}\left(\frac{10}{\sqrt{58}\sqrt{11}}\right)$ $= 66.68^o$	$\lvert\boldsymbol{f}\rvert = \sqrt{2^2 + 3^2 + 5^2}$ $= \sqrt{38}$ $\lvert\boldsymbol{g}\rvert = \sqrt{1^2 + (-7)^2 + 2^2}$ $= \sqrt{54}$ $\boldsymbol{f}.\boldsymbol{g}$ $= 2(1) + 3(-7) + 5(2)$ $= -9$ $\cos\theta = \frac{-9}{\sqrt{38}\sqrt{54}}$ $\theta = \cos^{-1}\left(\frac{-9}{\sqrt{38}\sqrt{54}}\right)$ $= (180 - 78.54)^o$ $= 101.46^o$

6.3 (c) I can identify perpendicular vectors using the scalar product.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** then $\boldsymbol{a}.\boldsymbol{b} = 0$ and vice versa.

Are $\boldsymbol{a} = \begin{pmatrix} 4 \\ -6 \\ 10 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ perpendicular?

$\boldsymbol{a}.\boldsymbol{b} = 4(1) - 6(4) + 10(2)$

$= 0$

Therefore $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are perpendicular.*

$\boldsymbol{c} = \begin{pmatrix} 3 \\ -2 \\ k \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$ are perpendicular. Find the value of k.

$\boldsymbol{c}$ *and* $\boldsymbol{d}$ *are perpendicular if* $\boldsymbol{c}.\boldsymbol{d} = 0$.

$3(4) - 2(5) + k(-2) = 0$

$2 - 2k = 0$

$2 = 2k$

$1 = k$

$\boldsymbol{s} = \begin{pmatrix} p \\ -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{t} = \begin{pmatrix} p \\ 7 \\ 2p \end{pmatrix}$ are perpendicular. Find the values of p.

$\boldsymbol{s}$ *and* $\boldsymbol{t}$ *are perpendicular if* $\boldsymbol{s}.\boldsymbol{t} = 0$.

$p(p) - 1(7) + 3(2p) = 0$

$p^2 + 6p - 7 = 0$

$(p + 7)(p - 1) = 0$

$p = -7, 1$

6.3 (d) I can use the commutative and distributive laws with vectors.

The **scalar product** can be used with more than two vectors by using the properties:

- For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$ this is the **commutative law**.
- For vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$ this is the **distributive law**.

$\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$. Calculate $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c})$.

$\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$

$= 2(4) - 3(0) + 1(2) + 2(-1) - 3(4) + 1(5)$

$= 1$

Vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ are such that $|\boldsymbol{c}| = 12, |\boldsymbol{d}| = 5$ and $\boldsymbol{c}.\boldsymbol{d} = 18$. Find $\boldsymbol{d}.(\boldsymbol{c} + \boldsymbol{d})$.

$\boldsymbol{d}.(\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{d}.\boldsymbol{c} + \boldsymbol{d}.\boldsymbol{d}$

$= 18 + |\boldsymbol{d}||\boldsymbol{d}| \cos\theta$

$= 18 + 5 \times 5 \times \cos 0^o$

$= 18 + 25$

$= 43$

$\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$. Calculate $\boldsymbol{b}.(\boldsymbol{a} + \boldsymbol{c})$.

$\boldsymbol{b}.(\boldsymbol{a} + \boldsymbol{c}) = \boldsymbol{b}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{c}$

$= 4(2) + 0(-3) + 2(1) + 4(-1) + 0(4) + 2(5)$

$= 16$

Vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are such that $|\boldsymbol{r}| = 7, |\boldsymbol{s}| = 10$ and $\boldsymbol{r}.\boldsymbol{s} = 20$. Find $\boldsymbol{r}.(\boldsymbol{r} + \boldsymbol{s})$.

$\boldsymbol{r}.(\boldsymbol{r} + \boldsymbol{s}) = \boldsymbol{r}.\boldsymbol{r} + \boldsymbol{r}.\boldsymbol{s}$

$= |\boldsymbol{r}||\boldsymbol{r}| \cos\theta + 20$

$= 7 \times 7 \times \cos 0^o + 20$

$= 49 + 20$

$= 69$

Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are such that $|\boldsymbol{u}| = 5, |\boldsymbol{v}| = 7$ and $\boldsymbol{u}.(\boldsymbol{u} + \boldsymbol{v}) = 50$. Find the angle between them.

$\boldsymbol{u}.\boldsymbol{u} + \boldsymbol{u}.\boldsymbol{v} = 50$

$|\boldsymbol{u}||\boldsymbol{u}| \cos\theta_1 + |\boldsymbol{u}||\boldsymbol{v}| \cos\theta_2 = 50$

$5 \times 5 \times \cos 0^o + 5 \times 7 \times \cos\theta = 50$

$25 + 35\cos\theta = 50$

$35\cos\theta = 25$

$\cos\theta = \frac{25}{35}$

$\theta = \cos^{-1}\left(\frac{25}{35}\right)$

$= 44.42^o$

7. Exponentials & Logarithms

7.1 Laws of Logarithms

7.1 (a) I can evaluate logarithmic expressions.

Earlier in the course you looked at the **exponential function**, $f(x) = a^x$, and its inverse the **logarithmic function,** $f(x) = \log_a x$. You can convert between these two forms which allows you to **evaluate logarithms**:

$$N = a^x \Leftrightarrow \log_a N = x$$

$\log_4 16$	$\log_5 125$	$\log_3 81$	$\log_7 \frac{1}{49}$	$\log_4 \frac{1}{64}$	$\log_2 \frac{1}{32}$	$\log_9 3$	$\log_{16} 2$	$\log_4 8$
$= 2$	$= 3$	$= 4$	$= -2$	$= -3$	$= -5$	$= 1/2$	$= 1/4$	$= 3/2$

7.1 (b) I can use the laws of logarithms to simplify expressions.

You can **simplify logarithmic expressions** using the following '**Laws of Logarithms**':

$$\log_a(bc) = \log_a b + \log_a c \qquad \log_a(b/c) = \log_a b - \log_a c \qquad \log_a(b^n) = n \log_a b$$

$$\log_a 1 = 0 \qquad \log_a a = 1$$

$\log_2 6 + \log_2 24 - 2\log_2 3.$	$2\log_6 4 + \frac{1}{3}\log_6 27 - \log_6 8.$	$(\log_d 25c^2) \div (\log_d 5c).$
$= \log_2 6 + \log_2 24 - \log_2 9$	$= \log_6 16 + \log_6 3 - \log_6 8$	$= (\log_b (5c)^2) \div (\log_b 5c)$
$= \log_6 144 - \log_2 9$	$= \log_6 48 - \log_6 8$	$= (2\log_b 5c) \div (\log_b 5c)$
$= \log_2 16$	$= \log_6 6$	$= 2$
$= 4$	$= 1$	

7.2 Logarithmic Equations

7.2 (a) I can solve equations involving logarithms.

Using the '**Laws of Logarithms**' you may solve **logarithmic equations**.

$2\log_x 6 - \log_x 3 = \frac{1}{2}$	$\log_2(x-4) - \log_2 3 = 4$	$\log_3(x+7) - \log_3(x-1) = 2$
$2\log_x 6 - \log_x 3 = \frac{1}{2}\log_x x$	$\log_2(x-4) - \log_2 3 = 4\log_2 2$	$\log_3(x+7) - \log_3(x-1) = 2\log_3 3$
$\log_x 36 - \log_x 3 = \log_x x^{\frac{1}{2}}$	$\log_2(x-4) - \log_2 3 = \log_2 16$	$\log_3(x+7) - \log_3(x-1) = \log_3 9$
$\log_x 12 = \log_x x^{\frac{1}{2}}$	$\log_2\left(\frac{x-4}{3}\right) = \log_2 16$	$\log_3\left(\frac{x+7}{x-1}\right) = \log_3 9$
$12 = x^{\frac{1}{2}}$	$\frac{x-4}{3} = 16$	$\frac{x+7}{x-1} = 9$
$144 = x$	$x - 4 = 48$	$x + 7 = 9x - 9$
		$16 = 8x$
		$2 = x$

7.2 (b) I can solve equations involving logarithms and factorisation.

Sometimes you may be given a **logarithmic equation** which will require the use of **factorisation**.

$\log_2(x-2)+\log_2(x-1)=1$

$\log_2(x-2)+\log_2(x-1) = \log_2 2$

$\log_2(x-2)(x-1)=\log_2 2$

$(x-2)(x-1)=2$

$x^2-3x+2=2$

$x^2-3x=0$

$x(x-3)=0$

$x=0,3$

$x=3$ *as* $(x-1)$ *and* $(x-2)$ *must be positive.*

$\log_{30}(x-2)+\log_{30}(x-1)=1$

$\log_{30}(x-2)+\log_{30}(x-1) = \log_{30} 30$

$\log_{30}(x-2)(x-1)=\log_{30} 30$

$(x-2)(x-1)=30$

$x^2-3x+2=30$

$x^2-3x-28=0$

$(x+4)(x-7)=0$

$x=-4,7$

$x=7$ *as* $(x-2)$ *and* $(x-1)$ *must be positive.*

$\log_2(x-1)+\log_2(x+1)=3$

$\log_2(x-1)+\log_2(x+1) = 3\log_2 2$

$\log_2(x-1)+\log_2(x+1) = \log_2 8$

$\log_2(x-1)(x+1)=\log_2 8$

$(x-1)(x+1)=8$

$x^2-1=8$

$x^2-9=0$

$(x+3)(x-3)=0$

$x=-3,3$

$x=3$ *as* $(x-1)$ *and* $(x+1)$ *must be positive.*

7.3 Exponential Growth

7.3 (a) I can solve problems on exponential growth and decay.

A **growth function** can represent appreciation, nuclear chain reactions, bacteria, viruses and population growth.

A **decay function** can represent depreciation, heat transfer, vibration and radioactivity.

The number e is an **irrational number** used frequently in mathematics and is approximately 2.718.

- Logarithms to the base e, $\log_e x$, are called **natural logarithms** and can be denoted $\ln x$**.**
- $\ln x$ and e^x are inverses of one another, i.e. $\ln e^x = e^{\ln x} = x$.

A rumour starts spreading around a school. The spread of the rumour can be modelled by $p = 500e^{0.05t}$ where p pupils know the rumour after t minutes.

($t = 0$ was when staff first became aware of the rumour spreading).

How many minutes till 1000 pupils hear the rumour?

$1000 = 500e^{0.05t}$

$e^{0.05t} = 2$

$0.05t = \ln 2$

$t = \dfrac{\ln 2}{0.05}$

$= 13.863$

$= 14$ *minutes*

The school bell rings and a thousand students head home.

The number of students, s, still in the school t minutes after the bell is modelled by $s = 1000 - 600\ln(t+1)$.

After how many minutes will the school be empty? Give your answer to 1 decimal place.

$0 = 1000 - 600\ln(t+1)$

$600\ln(t+1) = 1000$

$\ln(t+1) = \dfrac{5}{3}$

$t+1 = e^{\frac{5}{3}}$

$t = e^{\frac{5}{3}} - 1$

$= 4.294$

$= 4.3$ *minutes*

The mass M_t of a radioactive substance at time t years is given by the equation $M_t = M_0 e^{-kt}$ where M_0 is the initial mass.

90 g of this decays to leave 30 g over 4 years. Find k to 3 d.p.

$30 = 90e^{-4k}$

$\dfrac{1}{3} = e^{-4k}$

$\ln\dfrac{1}{3} = -4k$

$-\dfrac{\ln\frac{1}{3}}{4} = k$

$0.2747 = k$

$0.275 = k$

Another sample weighs 500 g, how much will remain in 20 years?

$M_{20} = 500e^{-0.275\times 20} = 2.04\ g$

7.4 Logarithmic Graphs

7.4 (a) I can work with exponential and logarithmic graphs.

You should be able to work with both **logarithmic** and **exponential functions**.

Note: $y = \log_a x$ passes through $(1, 0)$ and $(a, 1)$ while $y = a^x$ passes through $(0, 1)$ and $(1, a)$.

Given $y = \log_a(x - b)$ passes through (5, 0) and (10, 1) find a and b.

$y = \log_a x \text{ passes through } (1,\ 0) \text{ and } (a,\ 1).$

$y = \log_6(x - 4) \text{ passes through } (5,\ 0) \text{ and } (10,\ 1).$

$a = 6$

$b = 4$

Given $y = \log_a(x - b)$ passes through (2, 0) and (4, 1) find a and b.

$y = \log_a x \text{ passes through } (1,\ 0) \text{ and } (a,\ 1).$

$y = \log_3(x - 1) \text{ passes through } (2,\ 0) \text{ and } (4,\ 1).$

$a = 3$

$b = 1$

Given $y = \log_4(x - 3)$ passes through (p, 3) determine the value of p.

$When\ x = p,\ y = 3.$

$3 = \log_4(p - 3)$

$3\log_4 4 = \log_4(p - 3)$

$\log_4 64 = \log_4(p - 3)$

$64 = p - 3$

$61 = p$

Find the x-value where $y = \log_3(x - 2) - 4$ intersects the x-axis.

$On\ the\ x\text{-}axis\ y = 0.$

$0 = \log_3(x - 2) - 4$

$4 = \log_3(x - 2)$

$4\log_3 3 = \log_3(x - 2)$

$\log_3 81 = \log_3(x - 2)$

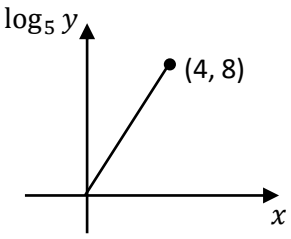
$81 = x - 2$

$83 = x$

7.4 (b) I can solve problems on experimental data resulting in graphs.

When **experimental data** is collected it is often necessary to find a relationship connecting the variables.

Express y in terms of x.



$m = \frac{8 - 0}{4 - 0}$

$= 2$

$\log_5 y = 2x$

$\log_5 y = 2x \log_5 5$

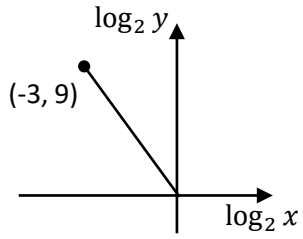
$\log_5 y = \log_5 5^{2x}$

$y = 5^{2x}$

$= (5^2)^x$

$= 25^x$

Express y in terms of x.



$m = \frac{9 - 0}{-3 - 0}$

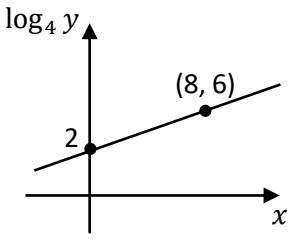
$= -3$

$\log_2 y = -3\log_2 x$

$\log_2 y = \log_2 x^{-3}$

$y = x^{-3}$

Express y in terms of x.



$m = \frac{6 - 2}{8 - 0}$

$= \frac{1}{2}$

$\log_4 y = \frac{1}{2}x + 2$

$\log_4 y = \left(\frac{1}{2}x + 2\right)\log_4 4$

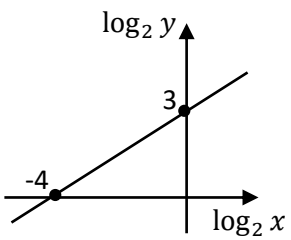
$\log_4 y = \log_4 4^{\frac{1}{2}x+2}$

$y = 4^{\frac{1}{2}x+2}$

$= 4^2\left(4^{\frac{1}{2}}\right)^x$

$= 16 \times 2^x$

Express y in terms of x.



$m = \frac{3 - 0}{0 - -4}$

$= \frac{3}{4}$

$\log_2 y = \frac{3}{4}\log_2 x + 3$

$\log_2 y$

$= \frac{3}{4}\log_2 x + 3\log_2 2$

$\log_2 y = \log_2 x^{\frac{3}{4}} + \log_2 8$

$\log_2 y = \log_e 8x^{\frac{3}{4}}$

$y = 8x^{\frac{3}{4}}$

8. Integration

8.1 Basic Integration

8.1 (a) I can use the basic rules of integration.

When you **integrate** a function you get its **integral**. **Integration** is the inverse process of **differentiation**.

To **integrate** a function $f(x) = ax^n$, make sure to remember the **constant of integration c.**

1. Increase the power by 1.
2. Divide the function by its power.

$$\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$$

$\int 2a^5 + a - 7 \, da$	$\int 3x^4 + 2x^{-\frac{1}{2}} - 6 \, dx$	$\int s^{\frac{3}{2}} - 7s \, ds$	$\int \frac{2}{7x^4} + 3 \, dx$
$= \frac{1}{3}a^6 + \frac{1}{2}a^2 - 7a + c$	$= \frac{3}{5}x^5 + 4x^{\frac{1}{2}} - 6x + c$	$= \frac{2}{5}x^{\frac{5}{2}} - \frac{7}{2}x^2 + c$	$= \int \frac{2}{7}x^{-4} + 3 \, dx$
			$= -\frac{2}{21}x^{-3} + 3x + c$

8.1 (b) I can use the basic rules of integration with products and quotients.

You can also find the integrals of **products** and **quotients** as the sum of individual terms.

$\int (2x+3)(x-2) \, dx.$	$\int \frac{3-4p^3}{p^2} dp$	$\int t(3t^2 - t + 5) \, dt$	$\int \frac{4x-1}{x^3} dx$
$= \int 2x^2 - x - 6 \, dx$	$= \int 3p^{-2} - 4p \, dp$	$= \int 3t^3 - t^2 + 5t \, dx$	$= \int 4x^{-2} - x^{-3} \, dx$
$= \frac{2}{3}x^3 - \frac{1}{2}x^2 - 6x + c$	$= -3p^{-1} + 2p^2 + c$	$= \frac{3}{4}t^4 - \frac{1}{3}t^3 + \frac{5}{2}t^2 + c$	$= -4x^{-1} + \frac{1}{2}x^{-2} + c$

8.1 (c) I can evaluate a definite integral.

A **definite integral** is one of the form:

$$\int_a^b f(x) \, dx.$$

Definite integrals do not require a **constant of integration**.

Evaluate the following:

$\int_0^2 x^3 + 2x^2 \, dx.$

$= \left[\frac{1}{4}x^4 + \frac{2}{3}x^3\right]_{-1}^{2}$

$= \left(\frac{1}{4}(2)^4 + \frac{2}{3}(2)^3\right) - (0)$

$= \frac{39}{4}$

Evaluate the following:

$\int_4^9 x^{-\frac{1}{2}} + 1 \, dx.$

$= \left[2x^{\frac{1}{2}} + x\right]_4^9$

$= \left(2(9)^{\frac{1}{2}} + (9)\right) - \left(2(4)^{\frac{1}{2}} + (4)\right)$

$= 7$

Find the value of t such that:

$\int_0^t 4x - 2 \, dx = 40.$

$[2x^2 - 2x]_0^t = 40$

$(2t^2 - 2t) - (0) = 40$

$2t^2 - 2t - 40 = 0$

$2(t^2 - t - 20) = 0$

$2(t-5)(t+4) = 0$

$t = -4, 5$

$t = 5 \; as \; -4 \; is \; negative$

8.2 Integration & Area

8.2 (a) I can calculate the area of a curve above the x-axis.

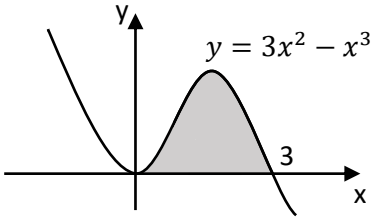
You can use **definite integrals** to work out the **area under a curve**.

The **area** bounded by curve $y = f(x)$, the x-axis and vertical lines $x = a$ and $x = b$ is:

$$\int_a^b f(x)\,dx$$

You may need to find the **limits** by finding the **points of intersection** of the curve and the x-axis.

Find the shaded area.



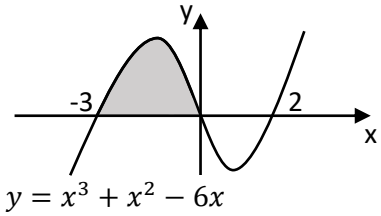
$$Area = \int_0^3 3x^2 - x^3\,dx$$

$$= \left[x^3 - \frac{1}{4}x^4\right]_0^3$$

$$= \left((3)^3 - \frac{1}{4}(3)^4\right) - (0)$$

$$= \frac{27}{4}\ units^2$$

Find the shaded area.



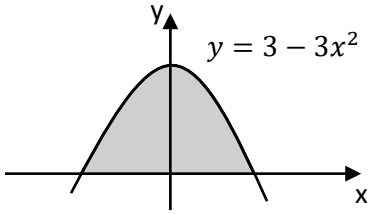
$$Area = \int_{-3}^0 x^3 + x^2 - 6x\ dx$$

$$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right]_{-3}^0$$

$$= (0) - \left(\frac{1}{4}(-3)^4 + \frac{1}{3}(-3)^3 - 3(-3)^2\right)$$

$$= \frac{9}{4}\ units^2$$

Find the shaded area.



$$3 - 3x^2 = 0$$

$$3(1 - x^2) = 0$$

$$3(1 + x)(1 - x) = 0$$

$$x = -1,\ 1$$

$$Area = 2\int_0^1 3 - 3x^2\ dx$$

$$= 2[3x - x^3]_0^1$$

$$= 2\big((3(1) - (1)^3) - (0)\big)$$

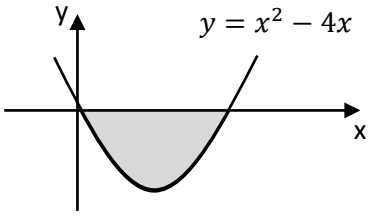
$$= 4\ units^2$$

8.2 (b) I can calculate the area of a curve both above and below the x-axis.

If you evaluate a **definite integral** above the x-axis the answer will be **positive** and below the x-axis, **negative**.

It is therefore important to find the **roots** of the function to divide the area into different parts.

Find the shaded area.

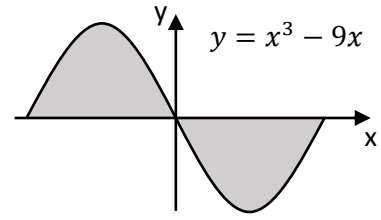


$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0, 4$$

Find the shaded area.



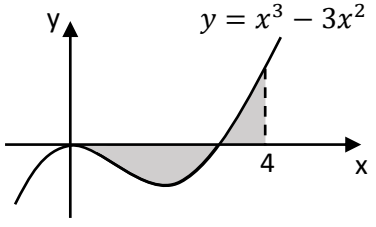
$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$$x(x + 3)(x - 3) = 0$$

$$x = -3, 0, 3$$

Find the shaded area.



$$x^3 - 3x^2 = 0$$

$$x^2(x - 3) = 0$$

$$x = 0, 3$$

$$Area = -\int_0^4 x^2 - 4x\,dx$$

$$= -\left[\frac{1}{3}x^3 - 2x^2\right]_0^4$$

$$= -\left(\frac{1}{3}(4)^3 - 2(4)^2\right) + (0)$$

$$= \frac{32}{3}\ units^2$$

$$Area = 2\int_{-3}^0 x^3 - 9x\,dx$$

$$= 2\left[\frac{1}{4}x^4 - \frac{9}{2}x^2\right]_{-3}^0$$

$$= 2\left((0) - \left(\frac{1}{4}(-3)^4 - \frac{9}{2}(-3)^2\right)\right)$$

$$= \frac{81}{2}\ units^2$$

$$Area = -\int_0^3 x^3 - 3x^2\ dx$$

$$= -\left[\frac{1}{4}x^4 - x^3\right]_0^3$$

$$= -\left(\frac{1}{4}(3)^4 - (3)^3\right) + (0)$$

$$= \frac{27}{4}\ units^2$$

$$Area = \left[\frac{1}{4}x^4 - x^3\right]_3^4$$

$$= \left(\frac{1}{4}(4)^4 - (4)^3\right) - \left(\frac{1}{4}(3)^4 - (3)^3\right)$$

$$= \frac{27}{4}\ units^2$$

$$Total\ Area = \frac{27}{4} \times 2 = \frac{27}{4}\ units^2$$

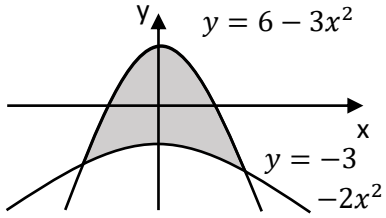
8.2 (c) I can calculate the area between two curves.

You can find the **area between two curves** $y = f(x)$ and $y = g(x)$, the vertical lines $x = a$ and $x = b$ where $f(x) \geq g(x)$ and $a \leq x \leq b$ using:

$$\int_a^b f(x) - g(x)\ dx$$

In a similar way to the last lesson it can be important to divide the area into different parts.

Find the shaded area.



$$6 - 3x^2 = -3 - 2x^2$$

$$9 - x^2 = 0$$

$$(3 + x)(3 - x) = 0$$

$$x = -3, 3$$

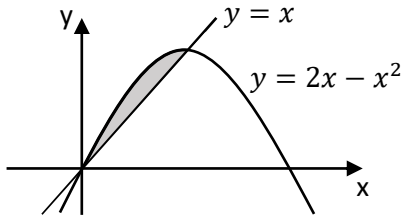
$$Area = 2\int_0^3 9 - x^2\,dx$$

$$= \left[9x - \frac{1}{3}x^3\right]_0^3$$

$$= 2\left(\left(9(3) - \frac{1}{3}(3)^3\right) - (0)\right)$$

$$= 36\ units^2$$

Find the shaded area.



$$2x - x^2 = x$$

$$x - x^2 = 0$$

$$x(1 - x) = 0$$

$$x = 0, 1$$

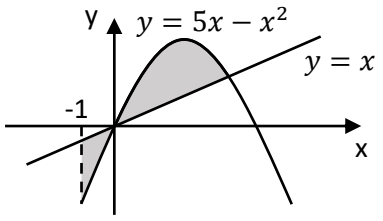
$$Area = \int_0^1 x - x^2\,dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \left(\frac{1}{2}(1)^2 - \frac{1}{3}(1)^3\right) - (0)$$

$$= \frac{1}{6}\ units^2$$

Find the shaded area.



$$5x - x^2 = x$$

$$4x - x^2 = 0$$

$$x(4 - x) = 0$$

$$x = 0, 3$$

$$Area = -\int_{-1}^0 4x - x^2\,dx$$

$$= -\left[2x^2 - \frac{1}{3}x^3\right]_{-1}^0$$

$$= -(0) + \left(2(-1)^2 - \frac{1}{3}(-1)^3\right)$$

$$= \frac{7}{3}\ units^2$$

		$Area = \left[2x^2 - \frac{1}{3}x^3\right]_0^3$ $= \left(2(3)^2 - \frac{1}{3}(3)^3\right) - (0)$ $= 9\ units^2$ $Total\ Area = \frac{7}{3} + 9 = \frac{34}{3}\ units^2$

8.3 Differential Equations

8.3 (a) I can evaluate the constant of integration in a differential equation.

A **differential equation** has the form:

$$\frac{dy}{dx} = f(x)$$

It can be **solved using integration**. This will leave a **constant of integration** which can only be solved with additional information.

Find $g(s)$ given that • $g'(s) = 2s + 4$ • $g(4) = 5.$ $g(s) = \int 2s + 4\ ds$ $= s^2 + 4s + c$ $g(4) = 4^2 + 4(4) + c$ $5 = 32 + c$ $-27 = c$ $g(s) = s^2 + 4s - 27$	Find $f(a)$ given that • $f'(a) = 3a^2 + 1$ • $f(2) = 24.$ $f(a) = \int 3a^2 - 7\ da$ $= a^3 - 7a + c$ $f(2) = 2^3 - 7(2) + c$ $24 = -6 + c$ $30 = c$ $f(a) = a^3 - 7a + 30$	Find $s(t)$ given that • $s'(t) = 2t - t^2$ • $s(3) = 7.$ $s(t) = \int 2t - t^2\ dt$ $= t^2 - \frac{1}{3}t^3 + c$ $s(3) = 3^2 - \frac{1}{3}(3)^3 + c$ $7 = c$ $s(t) = t^2 - \frac{1}{3}t^3 + 7$	Find $f(x)$ given that • $f'(x) = x^2 - 8$ • $f(3) = 10.$ $f(x) = \int x^2 - 8\ ds$ $= \frac{1}{3}x^3 - 8x + c$ $f(3) = \frac{1}{3}(3)^3 - 8(3) + c$ $10 = -15 + c$ $25 = c$ $f(x) = \frac{1}{3}x^3 - 8x + 25$

9. Trigonometric Formulae

9.1 Addition Formulae

9.1 (a) I can use addition formulae with exact values.

The following **addition formulae** appear on the formulae sheet in the exam.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Find $\sin(\pi + x)$.

$= \sin \pi \cos x + \cos \pi \sin x$

$= -\cos x$

Find $\cos(x + 30)^o$.

$= \cos x^o \cos 30^o - \sin x^o \sin 30^o$

$= \frac{\sqrt{3}}{2}\cos x^o + \frac{1}{2}\sin x^o$

Find $\sin 15^o$.

$= \sin(60 - 45)^o$

$= \sin 60^o \cos 45^o - \cos 60^o \sin 45^o$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

Find $\cos(\pi/12)$.

$= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

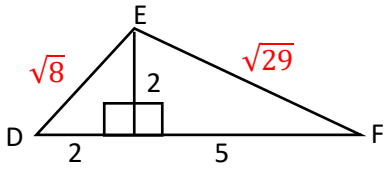
$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

9.1 (b) I can use addition formulae with triangles.

You may be given examples which rely on your knowledge of **Trigonometry** and **Pythagoras**.

Find the exact value of $\sin(d + f)$.



$DF = \sqrt{2^2 + 2^2} = \sqrt{8}$

$EF = \sqrt{2^2 + 5^2} = \sqrt{29}$

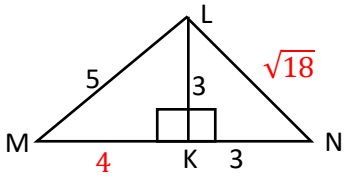
$\sin(d + f)$

$= \sin d \cos f + \cos d \sin f$

$= \left(\frac{2}{\sqrt{8}}\right)\left(\frac{5}{\sqrt{29}}\right) + \left(\frac{2}{\sqrt{8}}\right)\left(\frac{2}{\sqrt{29}}\right)$

$= \frac{14}{\sqrt{232}}$

Find the exact value of $\cos(m + n)$.



$MN = \sqrt{3^2 + 3^2} = \sqrt{18}$

$LK = \sqrt{5^2 - 3^2} = 4$

$\cos(m + n)$

$= \cos m \cos n - \sin m \sin n$

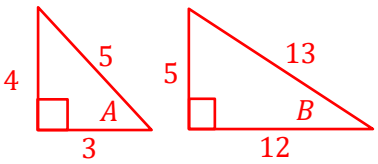
$= \left(\frac{4}{5}\right)\left(\frac{3}{\sqrt{18}}\right) + \left(\frac{3}{5}\right)\left(\frac{3}{\sqrt{18}}\right)$

$= \frac{21}{5\sqrt{18}}$

If A and B are two acute angles such that:

$$\sin A = \frac{4}{5}, \cos B = \frac{12}{13}$$

find the exact value of $\cos(A - B)$.



$\cos(A - B)$

$= \cos A \cos B - \sin A \sin B$

$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)$

$= \frac{56}{65}$

9.2 Double-Angles

9.2 (a) I can use the double-angle formulae with exact values.

The following **addition formulae** appear on the formulae sheet in the exam.

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

As there are **three versions** of the formula for $\cos 2A$, you must choose which is relevant.

Note: You can find $\sin 3A$ for instance by combining double-angle formulae and addition formulae.

Given $\sin x$ is $1/6$ find $\cos 2x$.

$= 1 - 2\sin^2 x$

$= 1 - 2\left(\frac{1}{6}\right)^2$

$= 1 - 2\left(\frac{1}{36}\right)$

$= 1 - \frac{1}{18}$

$= \frac{17}{18}$

Given $\cos x$ is $1/\sqrt{3}$ find $\cos 2x$.

$= 2\cos^2 x - 1$

$= 2\left(\frac{1}{\sqrt{3}}\right)^2 - 1$

$= 2\left(\frac{1}{3}\right) - 1$

$= \frac{2}{3} - 1$

$= -\frac{1}{3}$

Given $\sin x$ is $2/\sqrt{5}$ find $\cos 2x$.

$= 1 - 2\sin^2 x$

$= 1 - 2\left(\frac{2}{\sqrt{5}}\right)^2$

$= 1 - 2\left(\frac{4}{5}\right)$

$= 1 - \frac{8}{5}$

$= -\frac{3}{5}$

Find the exact value of $2\cos^2 15^o - 1$?

$= \cos 2(15)^o$

$= \cos 30^o$

$= \frac{\sqrt{3}}{2}$

Find the value of $2\cos^2 22.5^o - 1$?

$= \cos 2(22.5)^o$

$= \cos 45^o$

$= \frac{1}{\sqrt{2}}$

Find the exact value of $1 - 2\sin^2(\pi/8)$?

$= \cos 2(\pi/8)$

$= \cos(\pi/4)$

$= \frac{1}{\sqrt{2}}$

Find the value of $1 - 2\sin^2(\pi/12)$?

$= \cos 2(\pi/12)$

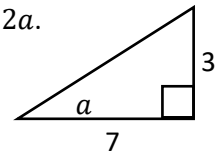
$= \cos(\pi/6)$

$= \frac{\sqrt{3}}{2}$

9.2 (b) I can use the double-angle formulae with triangles.

You may be given examples which rely on your knowledge of **Trigonometry** and **Pythagoras**.

Find the exact value of $\sin 2a$.



$\sqrt{7^2 + 3^2} = \sqrt{58}$

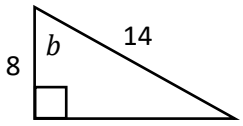
$\sin 2a = 2\sin a \cos a$

$= 2\left(\frac{3}{\sqrt{58}}\right)\left(\frac{7}{\sqrt{58}}\right)$

$= 2\left(\frac{21}{58}\right)$

$= \frac{21}{29}$

Find the value of $\cos 2b$?



$\cos 2b = 2\cos^2 x - 1$

$= 2\left(\frac{8}{14}\right)^2 - 1$

$= 2\left(\frac{8}{14}\right) - 1$

$= \frac{8}{7} - 1$

$= \frac{1}{7}$

Find the exact values of both $\cos x$ and $\sin x$ given that:

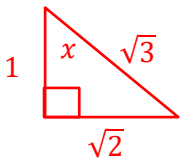
$\cos 2x = -\frac{1}{3}, 0 < x < \frac{\pi}{2}$.

$2\cos^2 x - 1 = -\frac{1}{3}$

$2\cos^2 x = \frac{2}{3}$

$\cos^2 x = \frac{1}{3}$

$\cos x = \frac{1}{\sqrt{3}}$



$\sqrt{(\sqrt{3})^2 - 1} = \sqrt{2}$

$\sin x = \frac{\sqrt{2}}{\sqrt{3}}$

Find the exact values of both $\cos x$ and $\sin x$ given that:

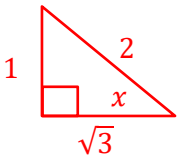
$\cos 2x = \frac{1}{2}, 0 < x < \frac{\pi}{2}$.

$2\cos^2 x - 1 = \frac{1}{2}$

$2\cos^2 x = \frac{3}{2}$

$\cos^2 x = \frac{3}{4}$

$\cos x = \frac{\sqrt{3}}{2}$



$\sqrt{2^2 - (\sqrt{3})^2} = 1$

$\sin x = \frac{1}{2}$

9.3 Wave Function

9.3 (a) I can write combinations of sine and cosine as single functions.

An expression of the form $a\cos x + b\sin x$ can be written as a **single function** i.e. $k\cos(x \pm \alpha)$ or $k\sin(x \pm \alpha)$:

$$k = \sqrt{a^2 + b^2} \qquad \tan\alpha = \frac{b}{a}$$

The angle α is called the **phase angle** and k is the **amplitude**.

Write $4\cos x^o + 3\sin x^o$ in the form $k\cos(x - \alpha)^o$, $0 \le \alpha \le 360$.

$k\cos(x-\alpha)^o$

$= k\cos x^o \cos\alpha^o + k\sin x^o \sin\alpha^o$

$= 4\cos x^o + 3\sin x^o$

$k\cos\alpha^o = 4$

$k\sin\alpha^o = 3$

$\tan\alpha^o = \frac{3}{4}$

$\alpha^o = \tan^{-1}\left(\frac{3}{4}\right)$

$= 36.87^o$

$(Q1\ as\ \sin +\ \&\ \cos +)$

$k = \sqrt{4^2 + 3^2}$

$= 5$

$4\cos x^o + 3\sin x^o = 5\cos(x - 36.87)^o$

Write $12\cos x - 5\sin x$ in the form $k\sin(x + \alpha)$, where $0 \le \alpha \le 2\pi$.

$k\sin(x+\alpha)$

$= k\sin x\cos\alpha + k\cos x\sin\alpha$

$= -5\sin x + 12\cos x$

$k\cos\alpha = -5$

$k\sin\alpha = 12$

$\tan\alpha = -\frac{12}{5}$

$\alpha = \tan^{-1}\left(-\frac{12}{5}\right)$

$= \pi - 1.18$

$(Q2\ as\ \sin +\ \&\ \cos -)$

$= 1.97$

$k = \sqrt{12^2 + (-5)^2}$

$= 13$

$12\cos x - 5\sin x = 13\sin(x + 1.97)$

Solve $k\sin a = -2$ & $k\cos a = -1$ where $k > 0$ & $0 \le a \le 360$.

$\tan a^o = 2$

$a^o = \tan^{-1}(2)$

$= (180 + 53.13)^o$

$(Q3\ as\ \sin -\ \&\ \cos -)$

$= 233.13^o$

$k = \sqrt{(-2)^2 + (-1)^2}$

$= \sqrt{5}$

Solve $k\sin a = 3$ & $k\cos a = -2$ where $k > 0$ & $0 \le a \le 360$.

$\tan a^o = -\frac{3}{2}$

$a^o = \tan^{-1}\left(-\frac{3}{2}\right)$

$= (180 - 56.31)^o$

$(Q3\ as\ \sin +\ \&\ \cos -)$

$= 123.69^o$

$k = \sqrt{3^2 + (-2)^2}$

$= \sqrt{13}$

9.3 (b) I can sketch the graph of a combination of two functions.

You can **sketch the graph** of a function $f(x) = a\cos x + b\sin x$, by expressing it as a **single function** of the form $k\cos(x \pm \alpha)$ or $k\sin(x \pm \alpha)$.

Similarly the **maxima** and **minima** can be found in the same way.

Function	$k\sin(x+\alpha)$	$k\cos(x+\alpha)$
Maxima of k	$x + \alpha = \frac{\pi}{2}$	$x + \alpha = 0, 2\pi$
Minima of k	$x + \alpha = \frac{3\pi}{2}$	$x + \alpha = \pi$

Note: For both **sine** and **cosine** further **maxima** and **minima** will occur at intervals of 2π.

Given $\sqrt{3}\sin x^o - \cos x^o$ can be written in the form $2\sin(x-30)^o$ find the points of intersection of $y = \sqrt{3}\sin x^o - \cos x^o$ and the x and y axes in the interval $0 \le x \le 2\pi$.

On the y-axis $x = 0$.

$y = 2\sin(0-30)^o$

$= -0.5$

Coordinates $(0, -0.5)$.

On the x-axis $y = 0$.

Occurs where:

$x - 30^o = 90^o$

$x = 120^o$

Coordinates $(120^o, 0)$.

Given $12\cos x - 5\sin x$ can be written as $\sqrt{74}\sin(x + 0.95)$ find the max and min values of the function and where they occur in the interval $0 \le x \le 2\pi$.

Maximum is $\sqrt{74}$.

Occurs where:

$x + 0.95 = \frac{\pi}{2}$

$x = 0.62$

Minimum is $-\sqrt{74}$.

Occurs where:

$x + 0.95 = \frac{3\pi}{2}$

$x = 3.76$

Given $4\cos x^o + 3\sin x^o$ can be written as $5\cos(x - 36.9)^o$ find the max and min values of the function and where they occur in the interval $0 \le x \le 360$.

Maximum is 5.

Occurs where:

$x - 36.9^o = 0^o$

$x = 36.9^o$

Minimum is -5.

Occurs where:

$x - 36.9^o = 180^o$

$x = 143.1^o$

Given that $\sin x - \cos x$ Can be written as

$$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

sketch the graph of $y = \sin x - \cos x$ for $0 \le x \le 2\pi$.

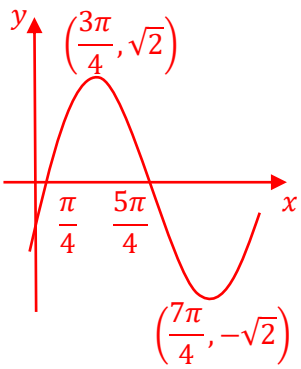
$y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

On the y-axis $x = 0$.

$y = \sqrt{2}\sin\left(0 - \frac{\pi}{4}\right)$

$= -1$

Coordinates $(0, -1)$.



9.4 Trig Applications

9.4 (a) I can solve equations involving the double-angle formulae.

You may be asked to solve equations involving **double-angle formulae** followed by **factorisation**.

Solve $3\sin 2x - 8\sin x = 0,\ 0 \le x \le 2\pi$.

$3(2\sin x\cos x) - 8\sin x = 0$

$6\sin x\cos x - 8\sin x = 0$

$2\sin x(3\cos x - 4) = 0$

$\sin x = 0$

$x = \sin^{-1}(0)$

$= 0, \pi, 2\pi$

$\cos x^o = \frac{4}{3}$

this has no solutions

Solve $3\cos 2x^o - \cos x^o = -1,\ 0 \le x \le 360$.

$3\cos 2x^o - \cos x^o = -1$

$3(2\cos^2 x^o - 1) - \cos x^o = -1$

$6\cos^2 x^o - 3 - \cos x^o = -1$

$6\cos^2 x^o - \cos x^o - 2 = 0$

$(2\cos x^o + 1)(3\cos x^o - 2) = 0$

$\cos x^o = -\frac{1}{2}$

$x^o = \cos^{-1}\left(-\frac{1}{2}\right)$

$= (180-60)^o, (180+60)^o$

$= 120^o, 240^o$

$\cos x^o = \frac{2}{3}$

$x^o = \cos^{-1}\left(\frac{2}{3}\right)$

$= 48.19^o,$

$(360 - 48.19)^o$

$= 48.19^o, 311.81^o$

9.4 (b) I can solve equations involving the wave function.

You may be asked to solve equations involving the **wave function**.

Given $4\cos x^o + 3\sin x^o$ can be written as $5\cos(x - 36.9)^o$ solve $4\cos x^o + 3\sin x^o = -1$ where $0 \leq x \leq 360$.

$$5\cos(x - 36.9)^o = -1$$

$$5\cos(x - 36.9)^o = -\frac{1}{5}$$

$$(x - 36.9)^o = \cos^{-1}\left(-\frac{1}{5}\right)$$

$$= (180 - 78.46)^o$$
$$(180 + 78.46)^o$$

$$= 101.54^o,\ 258.46^o$$

$$x = 138.44^o, 295.36^o$$

Given $3\cos x - 5\sin x$ can be written as $\sqrt{34}\sin(x - 2.60)$ solve $3\cos x - 5\sin x + 2 = 5$ where $0 \leq x \leq 2\pi$.

$$\sqrt{34}\sin(x - 2.60) + 2 = 5$$

$$\sqrt{34}\sin(x - 2.60) = 3$$

$$\sin(x - 2.60) = \frac{3}{\sqrt{34}}$$

$$x - 2.60 = \sin^{-1}\left(\frac{3}{\sqrt{34}}\right)$$

$$= 0.54,\ \pi - 0.54$$

$$= 0.54,\ 2.60$$

$$x = \pi,\ 5.20$$

Given $\cos 32t^o + 2\sin 32t^o$ can be written as $\sqrt{5}\cos(32t + 296.57)^o$ find the first solutions of $3 = 1 + \cos 32t^o + 2\sin 32t^o$.

$$3 = 1 + \cos 32t^o + 2\sin 32t^o$$

$$3 = 1 + \sqrt{5}\cos(32t + 296.57)^o$$

$$2 = \sqrt{5}\cos(32t + 296.57)^o$$

$$\frac{2}{\sqrt{5}} = \cos(32t + 296.57)^o$$

$$(32t + 296.57)^o = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$= 26.57^o, (360 - 26.57)^o$$

$$= 26.57^o, 333.43^o$$

$$32t^o = 36.86^o$$

$$t = 1.15^o$$

9.4 (c) I can use trigonometric identities in proofs.

From National 5: You should know the following:

There are two **trigonometric identities** which you are expected to know:

$$\tan x^o = \frac{\sin x^o}{\cos x^o}$$

$$\sin^2 x^o + \cos^2 x^o = 1$$

In Higher: You may have examples which involve **addition formulae** and **double-angle formulae**.

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

$$\frac{\sin^3 x + \sin x\cos^2 x}{\cos x} = \tan x$$

$$= \frac{\sin x\,(\sin^2 x + \cos^2 x)}{\cos x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$$\frac{\sin(a + b)}{\cos a\cos b} = \tan a + \tan b$$

$$= \frac{\sin a\cos b + \cos a\sin b}{\cos a\cos b}$$

$$= \frac{\sin a\cos b}{\cos a\cos b} + \frac{\cos a\sin b}{\cos a\cos b}$$

$$= \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}$$

$$= \tan a + \tan b$$

$$\frac{\sin 2x}{\tan x} = 2\cos^2 x$$

$$= \frac{2\sin x\cos x}{\tan x}$$

$$= \frac{2\sin x\cos x}{\frac{\sin x}{\cos x}}$$

$$= \frac{2\sin x\cos^2 x}{\sin x}$$

$$= 2\cos^2 x$$

10. Recurrence Relations

10.1 Terms in a Sequence

10.1 (a) I can evaluate the next term in a sequence.

The formula for a **linear recurrence relation** is:

$$u_{n+1} = au_n + b$$

Recurrence relations can be used to identify the **next terms in a sequence**.

Find u_2 when $u_0 = 5$ and $u_{n+1} = 3u_n - 6$.

$u_1 = 3(5) - 6$

$= 9$

$u_2 = 3(9) - 6$

$= 21$

Find v_6 when $v_4 = 16$ and $v_{n+1} = 0.5v_n + 4$.

$v_5 = 0.5(16) + 4$

$= 12$

$v_6 = 0.5(12) + 4$

$= 10$

Find t_9 when $t_7 = 2$ and $t_{n+1} = -4t_n + 1$.

$t_8 = -4(2) + 1$

$= -7$

$t_9 = -4(-7) + 1$

$= 29$

Find a_3 when $a_1 = 8$ and $a_{n+1} = 10 - 2a_n$.

$a_2 = 10 - 2(8)$

$= -6$

$a_3 = 10 - 2(-6)$

$= 22$

Find b_5 when $b_3 = 575$ and $b_{n+1} = 0.2b_n - 5$.

$b_4 = 0.2(575) - 5$

$= 110$

$b_5 = 0.2(110) - 5$

$= 17$

10.1 (b) I can evaluate the previous term in a sequence.

Recurrence relations can also be used to identify the **previous term in a sequence**.

Find u_1 when $u_3 = 25$ and $u_{n+1} = 2u_n + 3$.

$25 = 2u_2 + 3$

$22 = 2u_2$

$11 = u_2$

$11 = 2u_1 + 3$

$8 = 2u_1$

$4 = u_1$

Find v_5 when $v_7 = 2$ and $v_{n+1} = 6v_n - 4$.

$2 = 6v_6 - 4$

$6 = 6v_6$

$1 = 6v_6$

$1 = 6u_5 - 4$

$5 = 6u_5$

$\frac{5}{6} = u_5$

Find b_6 when $b_8 = 2$ and $b_n = 0.2b_{n-1} - 4$.

$2 = 0.2b_7 - 4$

$6 = 0.2b_7$

$30 = b_7$

$30 = 0.2b_6 - 4$

$34 = 0.2b_6$

$170 = b_6$

Find s_7 when $s_9 = 20$ and $s_n = -0.5s_{n-1} + 8$.

$20 = -0.5s_8 + 8$

$12 = 0.5s_8$

$24 = s_8$

$24 = 0.5s_7 + 8$

$16 = 0.5s_7$

$32 = s_7$

$a_{n+1} = 6a_n - 5$ and the difference between a_{k+1} and a_k is 45. Find a_k.

$a_{k+1} - a_k = 45$

$6a_k - 5 - a_k = 45$

$5a_k - 5 = 45$

$5a_k = 50$

$a_k = 10$

10.1 (c) I can find the formula of a recurrence relation from its sequence.

If you are given three terms of a sequence, you can find the formula for their **recurrence relation**. You can do this using **simultaneous equations**.

Find the formula for the recurrence relation with first terms 5, 15 and 35.

$u_1 = mu_0 + c \Rightarrow 15 = 5m + c$

$u_2 = mu_1 + c \Rightarrow 35 = 15m + c$

$20 = 10m$

$2 = m$

Find the formula for the recurrence relation with first terms 16, 12 and 10.

$u_1 = mu_0 + c \Rightarrow 12 = 16m + c$

$u_2 = mu_1 + c \Rightarrow 10 = 12m + c$

$2 = 4m$

$0.5 = m$

Find the formula for the recurrence relation with first terms 10, 23 and 62.

$u_1 = mu_0 + c \Rightarrow 23 = 10m + c$

$u_2 = mu_1 + c \Rightarrow 62 = 23m + c$

$39 = 13m$

$3 = m$

$m = 2 \Rightarrow 15 = 5(2) + c$ $15 = 10 + c$ $5 = c$ $u_{n+1} = 2u_n + 5$	$m = 0.5 \Rightarrow 12 = 16(0.5) + c$ $12 = 8 + c$ $4 = c$ $u_{n+1} = 0.5u_n + 4$	$m = 3 \Rightarrow 23 = 10(3) + c$ $23 = 30 + c$ $-7 = c$ $u_{n+1} = 3u_n - 7$

10.2 Limit of a Sequence

10.2 (a) I can identify whether a sequence has a limit and calculate this.

A sequence generated by a **recurrence relation** $u_{n+1} = au_n + b$ is:

- **Divergent** if it does not reach a limit.
- **Convergent** if it does reach a limit – this is only the case if $-1 < a < 1$. This means that as $n \to \infty$, the sequence will tend towards this limit.

The **limit** of a **convergent** sequence generated by a recurrence relation $u_{n+1} = au_n + b$ can be found using:

$$L = \frac{b}{1-a}$$

Does the recurrence relation $u_{n+1} = 0.2u_n - 4$ with $u_0 = 2$ have a limit? If so find this limit. *The sequence has a limit as* $-1 < 0.2 < 1$. $L = \frac{-4}{1-0.2}$ $= -\frac{4}{0.8}$ $= -5$	Does the recurrence relation $v_n = 0.9v_{n-1} + 3$ with $v_1 = 5$ have a limit? If so find this limit. *The sequence has a limit as* $-1 < 0.9 < 1$. $L = \frac{3}{1-0.9}$ $= \frac{3}{0.1}$ $= 30$	$u_{n+1} = (k-3)u_n + 2$ with $u_0 = 12$. For what values of k does this sequence have a limit? *The sequence will have a limit if*: $-1 < k - 3 < 1$ $2 < k < 4$	A sequence is defined by the recurrence relation $a_{n+1} = ka_n + 4$. If it tends to a limit of 8 determine the value of k. $L = \frac{4}{1-k}$ $8 = \frac{4}{1-k}$ $8(1-k) = 4$ $8 - 8k = 4$ $-8k = -4$ $k = 0.5$

10.2 (b) I can set up and solve recurrence relations in context.

Sometimes you will have to set up a **recurrence relation** before you can solve the problem.

A greenhouse is initially heated to 10°C above the outside temperature. It loses 25 % of its heat through the glass each hour. At the end of each hour a heater boosts its temperature by 2°C. Long term will the temperature stabilize?	A pedestrian precinct has 20 % of its litter cleared away each day by the work of the council cleansing department. 1.2 kg is dropped each day. The precinct appears tidy if no more than 5 kg is scattered around the precinct. Long term will this be the case?	A pond is treated weekly, this kills 68% of the bacteria present in the pond. Between the treatments, it is estimated that 600 million new bacteria appear. What will be the number of bacteria present in the pond long term?

$t_{n+1} = 0.75t_n + 2$ *The sequence will tend to a limit as* $-1 < 0.75 < 1$. *Therefore the temperature will stabilise.*	$u_{n+1} = 0.8u_n + 1.2$ $L = \frac{1.2}{1 - 0.8}$ $= \frac{1.2}{0.2}$ $= 6\ kg$ *The town will not appear tidy.*	$b_{n+1} = 0.32b_n + 600\ 000\ 000$ $L = \frac{600\ 000\ 000}{1 - 0.32}$ $= \frac{600}{0.68}$ $= 882\ 353\ 941\ bacteria$

11. The Circle

11.1 Centre & Radius

11.1 (a) I can find the equation of a circle given its centre and radius.

The equation of a circle with **centre** (a, b) and **radius** r is:

$$(x-a)^2+(y-b)^2=r^2$$

Find the centre and radius of the circle $(x+2)^2+(y-3)^2=144$. $C=(-2,\ 3)$ $r=\sqrt{144}=12$	Write the equation of the circle with centre (5, 1) and radius 3. $(x-5)^2+(y-1)^2=3^2$ $(x-5)^2+(y-1)^2=9$	Write the equation of the circle with centre (2, 9) and radius 1. $(x-2)^2+(y-9)^2=1^2$ $(x-2)^2+(y-9)^2=1$	A circle with centre A(3, 7) passes through B(3, 5). Find its equation. $r=\sqrt{0^2+2^2}=2$ $(x-3)^2+(y-7)^2=2^2$ $(x-5)^2+(y-1)^2=4$

11.2 Equation of a Circle

11.1 (a) I can find the equation of a circle given its centre and radius.

The **general equation** of a circle with **centre** $(-g,-f)$ and **radius** $\sqrt{g^2+f^2-c}$ is:

$$x^2+y^2+2gx+2fy+c=0$$

Find the centre and radius of the circle $x^2+y^2+8x-6y+16=0$. $g=4, f=-3, c=16$ $C=(-4,3)$ $r=\sqrt{4^2+(-3)^2-16}$ $=3$	Find the centre and radius of the circle $x^2+y^2-10x+24y=0$. $g=-5, f=12, c=0$ $C=(5,-12)$ $r=\sqrt{(-5)^2+12^2-0}$ $=13$	Find the centre and radius of the circle $x^2+y^2-8x+14y-3=0$. $g=-4, f=7, c=-3$ $C=(4,-7)$ $r=\sqrt{(-4)^2+7^2--3}$ $=\sqrt{68}$	Find k such that $x^2+y^2-8x+4ky+2=0$ has centre (4, 12). $g=-4, f=2k, c=2$ $C=(4,-4k)$ $-4k=12$ $k=-3$

11.2 (b) I can identify whether an equation represents a circle.

The **general equation** $x^2+y^2+2gx+2fy+c=0$ only defines a circle provided:

$$g^2+f^2-c>0$$

Explain why $x^2+y^2-6x+4y+14=0$ does not represent a circle. $g=-3, f=2, c=14$ $(-3)^2+2^2-14=-1$ <0 *Therefore not a circle.*	Does $x^2+y^2+6x-18y-1=0$ represent a circle? $g=3, f=-9, c=-1$ $3^2+(-9)^2-(-1)=91$ >0 *Therefore a circle.*	Does $x^2+y^2-8x+6y-7=0$ represent a circle? $g=-4, f=3, c=-7$ $(-4)^2+3^2-(-7)=32$ >0 *Therefore a circle.*	For what values of k does $x^2+y^2+2x-6y+k=0$ represent a circle? $g=1, f=-3, c=k$ $1^2+(-3)^2-k>0$ $10-k>0$ $10>k$

11.3 Lines & Circles

11.3 (a) I can find the equation of a tangent to a circle.

From National 5: You should know the following:

A **tangent** to a circle is **perpendicular** to the radius at the point of contact.

In Higher: This property can be used to find the **equation of a tangent to a circle:**

1. Find the **centre** of the circle.
2. Find the **gradient** of the line joining the centre to the given point.
3. Find the gradient **perpendicular** to this and substitute into $y - b = m(x - a)$.

A circle has centre (2, 6) and a tangent at (10, 10).

Find the equation of this tangent.

$$m = \frac{10-6}{10-2}$$
$$= \frac{1}{2}$$
$$m_{\perp} = -2$$
$$y - 10 = -2(x - 10)$$
$$y - 10 = -2x + 20$$
$$y = -2x + 30$$

A circle has centre (-1, 3) and a tangent at (4, 5).

Find the equation of this tangent.

$$m = \frac{-3-5}{1-4}$$
$$= \frac{8}{3}$$
$$m_{\perp} = -\frac{3}{8}$$
$$y - -3 = -\frac{3}{8}(x - 1)$$
$$8y + 24 = -3x + 3$$
$$8y = -3x - 21$$
$$y = -\frac{3}{8}x - \frac{21}{8}$$

The tangent of a circle at (5, 2) is $x + 2y = 6$. Find the radius of the circle going from the centre of the circle to (5, 2).

$$2y = -x + 6$$
$$y = -\frac{1}{2}x + 3$$
$$Therefore\ m = -\frac{1}{2}.$$
$$m_{\perp} = 2$$
$$y - 2 = 2(x - 5)$$
$$y - 2 = 2x - 10$$
$$y = 2x - 8$$

The tangent of a circle at (3, 6) is $x - 3y = 8$. Find the radius of the circle going from the centre of the circle to (3, 6).

$$-3y = -x + 8$$
$$y = \frac{1}{3}x - \frac{8}{3}$$
$$Therefore\ m = \frac{1}{3}.$$
$$m_{\perp} = -3$$
$$y - 6 = -3(x - 3)$$
$$y - 6 = -3x + 9$$
$$y = -3x + 15$$

11.3 (b) I can find the point(s) of intersection of a straight line and a circle.

You can find where a **straight line** and a **circle intersect** by solving their system of equations.

Find the point(s) of intersection of $y = 2x$ and the circle $(x-3)^2 + (y+1)^2 = 13$.

$$(x-3)^2 + ((2x)+1)^2 = 13$$
$$x^2 - 6x + 9 + 4x^2 + 4x + 1 = 13$$
$$5x^2 - 2x - 3 = 0$$
$$(5x+3)(x-1) = 0$$
$$x = 1, -0.6$$
$$x = 1 \Rightarrow y = 2(1)$$
$$= 2$$

Find the point(s) of intersection of $y = x + 1$ and the circle $x^2 + y^2 - 2x - 6y - 15 = 0$.

$$x^2 + (x+1)^2 - 2x - 6(x+1) - 15 = 0$$
$$x^2 + x^2 + 2x + 1 - 2x - 6x - 6 - 15 = 0$$
$$2x^2 - 6x - 20 = 0$$
$$2(x^2 - 3x - 10) = 0$$
$$2(x-5)(x+2) = 0$$
$$x = -2,\ 5$$

Find the point(s) of intersection of $y = 2x + 8$ and the circle $x^2 + y^2 + 4x + 2y - 20 = 0$.

$$x^2 + (2x+8)^2 + 4x + 2(2x+8) - 20 = 0$$
$$x^2 + 4x^2 + 32x + 64 + 4x + 4x + 16 - 20 = 0$$
$$5x^2 + 40x + 60 = 0$$
$$5(x^2 + 8x + 12) = 0$$
$$5(x+6)(x+2) = 0$$
$$x = -6, -2$$

$x = -0.6 \Rightarrow y = 2(-0.6)$ $= -1.2$ $Intersection\ (1,2), (-0.6, -1.2).$	$x = -2 \Rightarrow y = (-2) + 1$ $= -1$ $x = 5 \Rightarrow y = (5) + 1$ $= 6$ $Intersection\ (-2, -1), (5,\ 6).$	$x = -6 \Rightarrow y = 2(-6) + 8$ $= -4$ $x = -2 \Rightarrow y = 2(-2) + 8$ $= 4$ $Intersection\ (-6, -4), (-2,\ 4).$

11.3 (c) I can work with circles and tangents.

There are **three ways** a line and a circle can be associated:

- If the line cuts through the circle, there will be **two points of intersection** ($b^2 - 4ac > 0$).
- If the line is a **tangent** to the circle, there will be **one point of intersection** ($b^2 - 4ac = 0$).
- If the line misses the circle, there will be **no point of intersection** ($b^2 - 4ac < 0$).

Note: For a **tangent** you will also get **two equal solutions** when you solve the equation.

Show $y = 1$ is a tangent to circle $x^2 + y^2 + 20x + 18y + 81 = 0$ and find this point of intersection. $x^2 + 1^2 + 20x + 18(1) + 81 = 0$ $x^2 + 1 + 20x + 18 + 81 = 0$ $x^2 + 20x + 100 = 0$ $(x + 10)(x + 10) = 0$ $x = -10\ is\ a\ repeated\ root\ so$ $y = 1\ is\ a\ tangent.$ $Intersection\ (-10, 1).$	Show $y = x + 3$ is a tangent to the circle $x^2 + y^2 - 2y - 1 = 0$ and find this point of intersection. $x^2 + (x + 3)^2 - 2(x + 3) - 1 = 0$ $x^2 + x^2 + 6x + 9 - 2x - 6 - 1 = 0$ $2x^2 + 4x + 2 = 0$ $2(x^2 + 2x + 1) = 0$ $2(x + 1)(x + 1) = 0$ $x = -1\ is\ a\ repeated\ root\ so$ $y = x + 3\ is\ a\ tangent.$ $x = -1 \Rightarrow y = (-1) + 3$ $= 2$ $Intersection\ (-1,\ 2).$	Show $y = 2x + 8$ is a tangent to $x^2 + y^2 + 4x + 2y = 0$ and find this point of intersection. $x^2 + (2x + 8)^2 + 4x + 2(2x + 8) = 0$ $x^2 + 4x^2 + 32x + 64 + 4x + 4x + 16 = 0$ $5x^2 + 40x + 80 = 0$ $5(x^2 + 8x + 16) = 0$ $5(x + 4)(x + 4) = 0$ $x = -4\ is\ a\ repeated\ root\ so$ $y = 2x + 8\ is\ a\ tangent.$ $x = -4 \Rightarrow y = 2(-4) + 8$ $= 0$ $Intersection\ (-4,\ 0).$

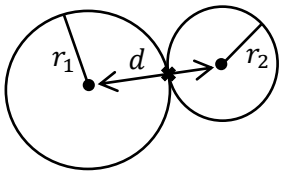
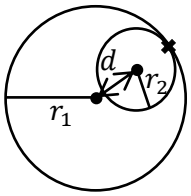
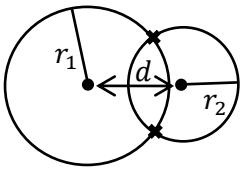
11.4 Intersecting Circles

11.4 (a) I can identify the type of intersection between two circles.

To investigate how two circles **intersect** you need both their **radii** and **centres** as well as the **distance** between their centres.

Note: Two circles are **congruent** if they are **identical**.

Circles can intersect in **three different ways**:

Externally	Internally	Twice
		
$d = r_1 + r_2$	$d = r_1 - r_2$	$r_1 - r_2 < d < r_1 + r_2$

Note: If $d > r_1 + r_2$ the two circles **do not intersect**.

Do the circles $(x-1)^2 + (y-1)^2 = 9$ and $(x-5)^2 + (y-4)^2 = 4$ intersect and if so in what way?

$C_1 = (1, 1)$

$r_1 = \sqrt{9}$

$= 3$

$C_2 = (5, 4)$

$r_2 = \sqrt{4}$

$= 2$

$d_{C_1C_2} = \sqrt{4^2 + 3^2}$

$= 5$

$r_1 + r_2 = 3 + 2$

$= 5$

$d_{C_1C_2} = r_1 + r_2$ *so they intersect externally.*

Do the circles $(x-2)^2 + (y-1)^2 = 5$ and $(x-4)^2 + (y-5)^2 = 45$ touch and if so, in what way?

$C_1 = (2, 1)$

$r_1 = \sqrt{5}$

$C_2 = (4, 5)$

$r_2 = \sqrt{45}$

$= 3\sqrt{5}$

$d_{C_1C_2} = \sqrt{2^2 + 4^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$

$r_2 - r_1 = 3\sqrt{5} - \sqrt{5}$

$= 2\sqrt{5}$

$d_{C_1C_2} = r_1 + r_2$ *so they intersect internally.*

Show that the two circles with equations $x^2 + y^2 - 6x - 4y + 9 = 0$ and $(x-4)^2 + (y-6)^2 = 26$ intersect.

$g = -3, f = -2, c = 9$

$C_1 = (3, 2)$

r_1

$= \sqrt{(-3)^2 + (-2)^2 - 9}$

$= 2$

$C_2 = (4, 6)$

$r_2 = \sqrt{26}$

$= 5.10$

$d_{C_1C_2} = \sqrt{1^2 + 4^2}$

$= 4.12$

$r_2 - r_1 = 5.10 - 2$

$= 3.10$

$r_2 + r_1 = 5.10 + 2$

$= 7.10$

$r_1 - r_2 < d_{C_1C_2} < r_1 + r_2$ *so they intersect at two places.*

Do the circles $x^2 + y^2 - 2x + 8y + 2 = 0$ and $x^2 + y^2 + 14x - 4y + 50 = 0$ intersect and if so in what way?

$g = -1, f = 4, c = 2$

$C_1 = (1, -4)$

$r_1 = \sqrt{(-1)^2 + 4^2 - 2}$

$= 3.87$

$g = 7, f = -2, c = 0$

$C_1 = (-7, 2)$

$r_1 = \sqrt{7^2 + 4^2 - 50}$

$= 3.87$

$d_{C_1C_2} = \sqrt{8^2 + 6^2}$

$= 10$

$r_1 + r_2 = 3.87 + 3.87$

$= 7.74$

$d_{C_1C_2} > r_1 + r_2$ *so they do not intersect.*

12. Further Calculus

12.1 Further Derivatives

12.1 (a) I can differentiate functions using the chain rule.

When you **integrate** a function you get its **integral**. **Integration** is the inverse process of **differentiation**.

$$f(x) = \big(g(x)\big)^n \Rightarrow f'(x) = n\big(g(x)\big)^{n-1} \times g'(x)$$

This can be remembered as:

1. Multiplying by the power.
2. Decreasing the power by 1.
3. Multiplying by the derivative of the terms inside the bracket.

Differentiate $y = (2x+5)^3$. $\frac{dy}{dx} = 3(2x+5)^2 \times 2$ $= 6(2x+5)^2$	Differentiate $s = (t^4-2)^5$. $\frac{ds}{dt} = 5(t^4-2)^4 \times 4t^3$ $= 20t^3(t^4-2)^4$	Differentiate $b = (7+2a)^{-3}$. $\frac{db}{da} = -3(7+2a)^{-4} \times 2$ $= -6(7+2a)^{-4}$	Differentiate $h(x) = \sqrt[3]{4x-7}$. $h(x) = (4x-7)^{\frac{1}{3}}$ $h'(x) = \frac{1}{3}(4x-7)^{-\frac{2}{3}} \times 4$ $= \frac{4}{3}(4x-7)^{-\frac{2}{3}}$

12.1 (b) I can differentiate functions involving sine and cosine.

You can differentiate **trigonometric functions** using:

$$f(x) = \sin x \Rightarrow f'(x) = \cos x \qquad f(x) = \cos x \Rightarrow f'(x) = -\sin x$$

Differentiate $y = 5\sin x$ and $s = 2\cos x$. $\frac{dy}{dx} = 5\cos x$ $\frac{ds}{dx} = -2\sin x$	Differentiate $y = 2\cos^3 x$. $\frac{dy}{dx} = 6\cos^2 x \times (-\sin x)$ $= -6\cos^2 x \sin x$	Differentiate $p = 2\sin^4 t$. $\frac{dp}{dt} = 8\sin^3 t \times (\cos t)$ $= 8\sin^3 t \cos t$	Differentiate $a = (\sin t + 7)^4$. $\frac{da}{dt} = 4(\sin t + 7)^3 \cos t$ $= 4\cos t\,(\sin t + 7)^3$

12.1 (c) I can differentiate functions involving sine, cosine and chain rule.

You can differentiate **trigonometric functions** involving **chain rule** using:

$$f(x) = \sin\big(g(x)\big) \Rightarrow f'(x) = \cos\big(g(x)\big) \times g'(x)$$
$$f(x) = \cos\big(g(x)\big) \Rightarrow f'(x) = -\sin\big(g(x)\big) \times g'(x)$$

Differentiate $f(x) = \cos 4x$ $f'(x) = -\sin 4x \times 4$ $= -4\sin 4x$	Differentiate $b = 2\cos(3t-4)$. $\frac{db}{dt} = -2\sin(3t-4) \times 3$ $= -6\sin(3t-4)$	Differentiate $a = 5\sin(2x+9)$. $\frac{da}{dx} = 5\cos(2x+9) \times 2$ $= 10\cos(2x+9)$	Differentiate $s = \sin(t^2+1)$. $\frac{ds}{dt} = \cos(t^2+1) \times 2t$ $= 2t\cos(t^2+1)$

12.2 Further Integrals

12.2 (a) I can integrate composite functions.

You can integrate **composite functions** using a variation of the **chain rule:**

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

This can be remembered as:

1. Increasing the power by 1.
2. Dividing by the (new) increased power.
3. Dividing by the derivative of the terms inside the bracket.

$\int (3x+1)^4 \, dx.$	$\int (5s-2)^5 \, ds.$	$\int \sqrt{2x+1} \, ds$	$\int \frac{1}{\sqrt[3]{(t-7)}} \, dt$
$= \frac{1}{5}(3x+1)^5 \times \frac{1}{3} + c$	$= \frac{1}{6}(5s-2)^6 \times \frac{1}{5} + c$	$= \int (2x+1)^{\frac{1}{2}} \, dx$	$= \int (7-t)^{-\frac{1}{3}} \, dt$
$= \frac{1}{15}(3x+1)^5 + c$	$= \frac{1}{30}(5s-2)^6 + c$	$= \frac{2}{3}(2x+1)^{\frac{3}{2}} \times \frac{1}{2} + c$	$= \frac{3}{2}(7-t)^{\frac{2}{3}} \times (-1) + c$
		$= \frac{1}{3}(2x+1)^{\frac{3}{2}} + c$	$= -\frac{3}{2}(7-t)^{\frac{2}{3}} + c$

12.2 (b) I can integrate functions involving sine and cosine.

You can integrate **trigonometric functions** involving **chain rule** using:

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c \qquad \int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$$

$\int \sin(4x-3) \, dx.$	$\int 7\cos(2x+9) \, dx.$	$\int 2\cos 3t + 3t^2 \, dt.$	$\int 5\sin 4t - 2t \, dt$
$= -\cos(4x-3) \times \frac{1}{4} + c$	$= 7\sin(2x+9) \times \frac{1}{2} + c$	$= -2\sin 3t \times \frac{1}{3} + t^3 + c$	$= 5\cos 4t \times \frac{1}{4} - t^2 + c$
$= -\frac{1}{4}\cos(4x-3) + c$	$= \frac{7}{2}\sin(2x+9) + c$	$= -\frac{2}{3}\sin 3t + t^3 + c$	$= \frac{5}{4}\cos 4t - t^2 + c$